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INDEX.

G. Darboux

The Motion of a Solid in Infinite Liquid under no Forces.

BY A. G. GREENHILL, F. R. S., *Woolwich, England.*

The theory is sketched out in Thomson and Tait's "Natural Philosophy," and a complete solution for a solid of revolution is given in Kirchhoff's "Vorlesungen über Mathematische Physik," IX; the treatment is developed at length in the "Motion of a Solid in a Liquid," by Dr. Thomas Craig (Van Nostrand, 1878), and in Halphen's "Fonctions elliptiques," t. II, chap. IV, and the subject in general has attracted the attention of a large number of writers, for which the Fortschritte der Mathematik may be consulted.

The object of the present paper is to examine closely the elliptic function expression of all the dynamical quantities involved, and to explore the analytical field by working out completely the simplest Pseudo-Elliptic cases to serve as landmarks, utilizing for this purpose the analysis developed in the paper on "Pseudo-Elliptic Integrals and their Dynamical Applications," Proc. London Math. Society, vol. XXV, and carrying this out in continuation of the manner employed for a similar purpose in the papers on the "Dynamics of a Top" and on the "Associated Motion of a Top and of a Body under no Forces," Proc. London Math. Society, vols. XXVI, XXVII.

The lectures delivered last year by Professor Klein at Princeton University have placed the analytical treatment of the motion of the top, and of Jacobi's two allied motions *à la Poinsot*, in a much clearer light and in a more elegant manner; a similar treatment of the present problem will certainly prove equally valuable, but meanwhile the special cases, developed at length here, will serve as oases, so to speak, in the infinite region of the general elliptic function solution.

Simple experimental illustrations of the motion can be observed in the evolutions of a plate or coin or bubble in water, or of a disc of paper or cardboard in the air, as well as in the motion of a projectile or torpedo.

1. The notation employed is that given in Basset's "Hydrodynamics," vol. I, Appendix III, and also in the "Applications of Elliptic Functions," p. 342; the body may for simplicity be taken as a smooth homogeneous solid of revolution, having component linear and angular velocities u, v, w and p, q, r; and now the total kinetic energy T of the body and of the surrounding infinite frictionless liquid stirred up by the motion of the body is given by an expression of the form

$$T = \tfrac{1}{2} P (u^2 + v^2) + \tfrac{1}{2} Rw^2 + \tfrac{1}{2} A (p^2 + q^2) + \tfrac{1}{2} Cr^2, \tag{A}$$

where P, R, A, C are constants depending on the shape of the body and on the density of the solid and liquid.

The more general form of T assumed by Halphen, due to certain modifications in the shape of the body, which still leads to Elliptic Function solutions, may be considered separately in its modification of the results, as also the effect of the *circulation* of the liquid, which may exist with a ring-shaped or perforated body.

To realize practically the condition that no external forces act upon the body, even in a field of gravity, we may make the density of the body and of the liquid the same, so that the buoyancy and weight cancel, and the apparent weight is zero, as in the case of a fish and a submarine boat or torpedo; and now the Hamiltonian equations of motion lead to

$$P\frac{du}{dt} - rPv + qRw = 0, \tag{1}$$

$$P\frac{dv}{dt} - pRw + rPu = 0, \tag{2}$$

$$R\frac{dw}{dt} - qPu + pPv = 0, \tag{3}$$

$$A\frac{dp}{dt} - rAq + qCr - wPv + vRw = 0, \tag{4}$$

$$A\frac{dq}{dt} - pCr + rAp - uRw + wPu = 0, \tag{5}$$

$$C\frac{dr}{dt} - qAp + pAq - vPu + uPv = 0; \tag{6}$$

some of the equations being capable of obvious simplifications.

2. To make the memoir complete, it will be advisable to repeat to a certain extent the ordinary treatment; thus, multiplying (1) by Pu, (2) by Pv, (3) by Rw, and adding,

$$P^2u\frac{du}{dt}+P^2v\frac{dv}{dt}+R^2w\frac{dw}{dt}=0, \tag{7}$$

and integrating,

$$P^2(u^2+v^2)+R^2w^2=F^2, \tag{B}$$

where the constant F represents the resultant *linear momentum* of the system.

Similarly it can be shown that

$$AuPp+AvPq+CwRr=G, \tag{C}$$

where G is a constant, representing the resultant *angular momentum* of the system.

From equations (A) and (B),

$$\begin{aligned} A(p^2+q^2) &= 2T-Cr^2-Rw^2-P(u^2+v^2) \\ &= 2T-Cr^2-\frac{F^2}{R}+F^2\left(\frac{1}{R}-\frac{1}{P}\right)(F^2-R^2w^2), \end{aligned}$$

and from equation (3),

$$\begin{aligned} R^2\frac{dw^2}{dt^2} &= P^2(uq-vp)^2 \\ &= P^2(u^2+v^2)(p^2+q^2)-P^2(up+vq)^2 \\ &= \frac{F^2}{A}\left(\frac{1}{R}-\frac{1}{P}\right)(F^2-R^2w^2)^2 \\ &\quad+\left(2T-Cr^2-\frac{F^2}{R}\right)\frac{F^2-R^2w^2}{A}-\left(\frac{G-CwRr}{A}\right)^2, \end{aligned} \tag{D}$$

thus determining w or Rw as an elliptic function of t.

3. By the Principles of the Conservation of Energy and Momentum, T will remain constant, while F will represent a constant linear momentum in a fixed direction, Oz suppose; so that denoting by γ_1, γ_2, γ_3 the cosines of the angles between Oz and the axes OA, OB, OC fixed in the body,

$$Pu=F\gamma_1,\quad Pv=F\gamma_2,\quad Rw=F\gamma_3; \tag{8}$$

and, introducing Euler's angles θ, ϕ, ψ,

$$\gamma_1 = -\sin\theta\cos\phi,\quad \gamma_2 = \sin\theta\sin\phi,\quad \gamma_3 = \cos\theta; \tag{9}$$

$$\begin{aligned} p &= \sin\phi\,\dot\theta - \sin\theta\cos\phi\,\dot\psi,\\ q &= \cos\phi\,\dot\theta + \sin\theta\sin\phi\,\dot\psi,\\ r &= \dot\phi + \cos\theta\,\dot\psi, \end{aligned} \tag{10}$$

so that

$$\begin{aligned} P(up + vq) &= F\sin\theta(-p\cos\phi + q\sin\phi)\\ &= F\sin^2\theta\,\frac{d\psi}{dt}, \end{aligned} \tag{11}$$

or

$$\frac{d\psi}{dt} = \frac{G - CrF\cos\theta}{AF\sin^2\theta}. \tag{12}$$

Split up into two partial fractions,

$$\frac{d\psi}{dt} = \frac{d\psi_1}{dt} + \frac{d\psi_2}{dt}, \tag{13}$$

where

$$\begin{aligned} \frac{d\psi_1}{dt} &= \frac{G - CrF}{2AF}\,\frac{1}{1-\cos\theta},\\ \frac{d\psi_2}{dt} &= \frac{G + CrF}{2AF}\,\frac{1}{1+\cos\theta}; \end{aligned} \tag{14}$$

and then

$$\begin{aligned} \frac{d\phi}{dt} &= r - \cos\theta\,\frac{d\psi}{dt}\\ &= \left(1 - \frac{C}{A}\right)r - \frac{d\psi_1}{dt} + \frac{d\psi_2}{dt}. \end{aligned} \tag{15}$$

4. Writing z for $\cos\theta$, or Fz for Rw in equation (D), then

$$\frac{dz^2}{dt^2} = n^2\left\{a(z^2-1)^2 - \left(2T - Cr^2 - \frac{F^2}{R}\right)\frac{z^2-1}{An^2} - \left(\frac{CrFz - G}{AnF}\right)^2\right\}, \tag{16}$$

where

$$an^2 = \frac{F^2}{A}\left(\frac{1}{R} - \frac{1}{P}\right); \tag{17}$$

and according as the body is prolate or oblate, so is $P > R$ or $P < R$, or an^2 is positive or negative.

To distinguish these two cases, we take $a = +1$ for prolate bodies and $a = -1$ for oblate bodies; so that

$$n^2 = \frac{F^2}{A}\left(\frac{1}{R} \sim \frac{1}{P}\right), \tag{18}$$

and now
$$\frac{dz}{dt}=n\surd Z, \tag{19}$$
where
$$Z=a(z^2-1)^2-\left(2T-Cr^2-\frac{F^2}{R}\right)\frac{z^2-1}{An^2}-\left(\frac{CrFz-G}{AnF}\right)^2 \tag{20}$$
and
$$\frac{d\psi}{dz}=\frac{G-Crz}{AnF(1-z^2)\surd Z}. \tag{21}$$

Following the notation employed in the discussion of the motion of a top, it is convenient to put
$$2T-Cr^2-\frac{F^2}{R}=An^2D, \tag{22}$$
and
$$D-\frac{G^2-C^2r^2F^2}{A^2n^2F^2}=E, \tag{23}$$
so that
$$Z=a(z^2-1)(z^2-1-aD)-\left(\frac{CrFz-G}{AnF}\right)^2, \tag{24}$$
$$Z=a(z^2-1)(z^2-1-aE)-\left(\frac{Gz-CrF}{AnF}\right)^2. \tag{25}$$

5. Starting now with the general elliptic integral of the first kind,
$$u=\int\frac{dz}{\surd Z}, \tag{26}$$
where the quartic
$$Z=az^4+4bz^3+6cz^2+4dz+e, \tag{27}$$

and u denotes the elliptic argument (a different use of the letter u to that employed previously to denote a component velocity; these two meanings will not however be found to clash); we make use of a theorem, supposed to be due to Weierstrass, but first employed by Biermann in his Dissertation, "Problemata quædam mechanica functionum ellipticarum ope soluta," 1865, which asserts that we can put
$$\wp(u_1\pm u_2)=\frac{F(z_1,z_2)\mp\surd Z_1\surd Z_2}{2(z_1-z_2)^2}, \tag{E}$$
where
$$F(z_1,z_2)=az_1^2z_2^2+2bz_1z_2(z_1+z_2)+c(z_1^2+4z_1z_2+z_2^2)+2d(z_1+z_2)+e, \tag{28}$$

the *second polar* of Z; u_1 and u_2 denoting the elliptic arguments corresponding to z_1 and z_2.

6. When the roots e_α, e_β, e_γ of the discriminating cubic

$$4e^3 - g_2 e - g_3 = 0 \tag{29}$$

are known, in which g_2 and g_3 are the *quadrinvariant* and *cubinvariant* of the quartic Z, given by

$$g_2 = ae - 4bd + 3c^2, \tag{30}$$

$$g_3 = ace + 2bcd - ad^2 - eb^2 - c^3, \tag{31}$$

the above theorem (E) can be replaced by

$$\wp(u_1 \pm u_2) - e_\alpha$$
$$= a\left\{\frac{\sqrt{(z_1 - z_0 . z_1 - z_\alpha)}\sqrt{(z_2 - z_\beta . z_2 - z_\gamma)} \mp \sqrt{(z_2 - z_0 . z_2 - z_\alpha)}\sqrt{(z_1 - z_\beta . z_1 - z_\gamma)}}{2(z_1 - z_2)}\right\}^2, \tag{32}$$

where z_0, z_α, z_β, z_γ denote the roots of the quartic $Z = 0$, so that

$$Z = a(z - z_0)(z - z_\alpha)(z - z_\beta)(z - z_\gamma). \tag{33}$$

With the form of Z in (20),

$$z_0 + z_\alpha + z_\beta + z_\gamma = 0, \tag{34}$$

$$a(z_0 + z_\alpha)(z_\beta + z_\gamma) + a(z_0 z_\alpha + z_\beta z_\gamma) = -2a - D - \frac{C^2 r^2}{A^2 n^2}, \tag{35}$$

$$a z_0 z_\alpha z_\beta z_\gamma = a + D - \frac{G^2}{A^2 n^2 F^2}. \tag{36}$$

Thence, denoting by v_1 and v_2 the values of the elliptic argument corresponding to

$$z_1 = +1 \text{ and } z_2 = -1,$$

$$Z_1 Z_2 = -\frac{C^2 r^2 F^2 - G^2}{A^2 n^2 F^2}, \tag{37}$$

$$\wp(v_1 \pm v_2) - e_\alpha$$
$$= \tfrac{1}{8} a\{1 - (z_0 + z_\alpha)(z_\beta + z_\gamma) + z_0 z_\alpha + z_\beta z_\gamma + z_0 z_\alpha z_\beta z_\gamma\} \pm \tfrac{1}{8}\frac{C^2 r^2 F^2 - G^2}{A^2 n^2 F^2}$$
$$= \tfrac{1}{8}\left\{a + 2a(z_0 + z_\alpha)^2 - 2a - D - \frac{C^2 r^2}{A^2 n^2} + a + D - \frac{G^2}{A^2 n^2 F^2} \pm \frac{C^2 r^2 F^2 - G^2}{A^2 n^2 F^2}\right\}; \tag{38}$$

or

$$\wp(v_1 + v_2) - e_\alpha = \tfrac{1}{4} a(z_0 + z_\alpha)^2 - \frac{G^2}{4A^2 n^2 F^2}, \tag{39}$$

$$\wp(v_1 - v_2) - e_\alpha = \tfrac{1}{4} a(z_0 + z_\alpha)^2 - \frac{C^2 r^2}{4A^2 n^2}; \tag{40}$$

so that
$$\tfrac{1}{4} a (z_0 + z_a)^2 = \frac{G^2}{4A^2n^2F^2} + \wp(v_1 + v_2) - e_a, \tag{41}$$
$$= \frac{C^2r^2}{4A^2n^2} \quad + \wp(v_1 - v_2) - e_a. \tag{42}$$

7. It is convenient to employ the notation of Darboux in the corresponding motion of the top (Proc. London Math. Society, vol. XXVII), and to put
$$\frac{G}{AnF} = \frac{2L}{M}, \quad \frac{Cr}{An} = \frac{2B}{M}; \tag{43}$$
so that
$$Z = a(z^2 - 1)(z^2 - 1 - aD) - 4\left(\frac{Bz - L}{M}\right)^2, \tag{44}$$
$$= a(z^2 - 1)(z^2 - 1 - aE) - 4\left(\frac{Lz - B}{M}\right)^2, \tag{45}$$
and
$$\frac{d\psi}{dz} = 2\frac{L - Bz}{M(1 - z^2)\sqrt{Z}}. \tag{46}$$
Also
$$\wp u - e_a = \frac{s - s_a}{M^2}, \tag{47}$$

where M is a *homogeneity factor* to be determined in the sequel, and s is a new variable defined by
$$\frac{u}{M} = \int\frac{ds}{\sqrt{S}} = \frac{1}{M}\int\frac{dz}{\sqrt{Z}}, \tag{48}$$
where
$$S = 4s(s + x)^2 - \{(y + 1)s - xy\}^2, \tag{49}$$
x and y being the quantities defined by Halphen in his "Fonctions elliptiques," t. I, p. 103; and when resolved into factors, we put
$$S = 4(s - s_a)(s - s_\beta)(s - s_\gamma), \text{ or } 4(s - s_1)(s - s_2)(s - s_3). \tag{50}$$

8. Now putting
$$v_1 + v_2 = v, \tag{51}$$
(again a different signification of v to that employed at the outset) and denoting by σ or $s(v)$ the value of s or $s(u)$ corresponding to $u = v$, equation (41) becomes
$$\tfrac{1}{4} aM^2(z_0 + z_a)^2 = L^2 + \sigma - s_a$$
$$= aN_a^2, \tag{52}$$
suppose; so that
$$\tfrac{1}{2} M(z_0 + z_1) = N_1,$$
$$\tfrac{1}{2} M(z_0 + z_2) = N_2,$$
$$\tfrac{1}{2} M(z_0 + z_3) = N_3. \tag{53}$$

Taking $s_1 > s_2 > s_3$, and employing relation (34), we find that when $a = +1$, as for a prolate solid,

$$
\begin{aligned}
Mz_0 &= -N_1 + N_2 + N_3, \\
Mz_3 &= +N_1 - N_2 + N_3, \\
Mz_2 &= +N_1 + N_2 - N_3, \\
Mz_1 &= -N_1 - N_2 - N_3;
\end{aligned} \tag{54}
$$

while for $a = -1$, and an oblate solid,

$$
\begin{aligned}
Mz_3 &= +N_1 + N_2 - N_3, \\
Mz_2 &= +N_1 - N_2 + N_3, \\
Mz_1 &= -N_1 + N_2 + N_3, \\
Mz_0 &= -N_1 - N_2 - N_3.
\end{aligned} \tag{55}
$$

9. Now, rewriting the expressions for Z, in this new notation,

$$
\begin{aligned}
Z &= a(z - z_0)(z - z_\alpha)(z - z_\beta)(z - z_\gamma) \\
&= a\left\{z^2 - 2\frac{N_\alpha}{M}z + \frac{N_\alpha^2 - (N_\beta - N_\gamma)^2}{M^2}\right\}\left\{z^2 + 2\frac{N_\alpha}{M} + \frac{N_\alpha^2 - (N_\beta - N_\gamma)^2}{M^2}\right\} \\
&= a\left(z^4 - 2\frac{N_1^2 + N_2^2 + N_3^2}{M^2}z^2 + 8\frac{N_1N_2N_3}{M^3}z\right. \\
&\qquad\left. + \frac{N_1^4 + N_2^4 + N_3^4 - 2N_2^2N_3^2 - 2N_3^2N_1^2 - 2N_1^2N_2^2}{M^4}\right);
\end{aligned} \tag{56}
$$

so that, putting $z = \pm 1$ in (56) and (44), (45),

$$
a\left(1 - 2\frac{N_1^2 + N_2^2 + N_3}{M^2} + 8\frac{N_1N_2N_3}{M^3} + \frac{N_1^4 + \ldots - 2N_2^2N_3^2 \ldots}{M^4}\right) = -4\left(\frac{B-L}{M}\right)^2, \tag{57}
$$

$$
a\left(1 - 2\frac{N_1^2 + N_2^2 + N_3^2}{M^2} - 8\frac{N_1N_2N_3}{M^3} + \frac{N_1^4 + \ldots - 2N_2^2N_3^2 \ldots}{M^4}\right) = -4\left(\frac{B+L}{M}\right)^2. \tag{58}
$$

Subtracting,

$$
16a\frac{N_1N_2N_3}{M^3} = 16\frac{BL}{M^2}
$$

or

$$
BLM = aN_1N_2N_3, \tag{59}
$$

so that

$$
\frac{Cr}{An} = \frac{2B}{M} = 2a\frac{N_1N_2N_3}{LM^2}, \tag{60}
$$

with

$$
\frac{G}{AnF} = \frac{2L}{M}. \tag{61}
$$

Multiplying equations (57) and (58) by a, and adding,

$$1-2\frac{N_1^2+N_2^2+N_3^2}{M^2}+\frac{N_1^4+N_2^4+N_3^4-2N_2^2N_3^2-2N_3^2N_1^2-2N_1^2N_2^2}{M^4}$$

$$=-4a\frac{B^2+L^2}{M^2}$$

$$=-4a\frac{N_1^2N_2^2N_3^2}{L^2M^4}-4a\frac{L^2}{M^2},$$

$$L^2M^4-2(N_1^2+N_2^2+N_3^2-2aL^2)L^2M^2$$
$$+(N_1^4+\ldots\ldots-2N_2^2N_3^2-\ldots\ldots)L^2+4aN_1^2N_2^2N_3^2=0,$$

$$L^2(M^2-N_1^2-N_2^2-N_3^2+2aL^2)^2=4(L^2-aN_1^2)(L^2-aN_2^2)(L^2-aN_3^2)$$
$$=4(s_1-\sigma)(s_2-\sigma)(s_3-\sigma)=-\Sigma, \qquad (62)$$

where Σ denotes the value of S in (50) when $s=\sigma$; and thus

$$M^2=N_1^2+N_2^2+N_3^2-2aL^2-\frac{\sqrt{-\Sigma}}{L}, \qquad (63)$$

thus determining the homogeneity factor M.

10. Equating the coefficients of z^2,

$$2a+D+4\frac{B^2}{M^2}=2a+E+4\frac{L^2}{M^2}$$
$$=2a\frac{N_1^2+N_2^2+N_3^2}{M^2}; \qquad (64)$$

so that
$$E=2a\frac{N_1^2+N_2^2+N_3^2-2aL^2-M^2}{M^2}$$
$$=2a\frac{\sqrt{-\Sigma}}{LM^2}; \qquad (65)$$

and
$$D=2a\frac{\sqrt{-\Sigma}}{LM^2}+4\frac{L^2-B^2}{M^2}. \qquad (66)$$

But otherwise, equating the coefficients of z^0,

$$a+D-4\frac{L^2}{M^2}=a+E-4\frac{B^2}{M^2}$$
$$=a\frac{N_1^4+\ldots\ldots-2N_2^2N_3^2\ldots\ldots}{M^4}, \qquad (67)$$

$$a+D=\frac{4L^2(N_1^2+N_2^2+N_3^2-2aL^2)-4L\sqrt{-\Sigma}+a(N_1^4+\ldots\ldots-2N_2^2N_3^2\ldots)}{M^4} \qquad (68)$$

and

$$N_1^2 + N_2^2 + N_3^2 = 3aL^2 + 3a\wp v, \tag{69}$$

$$N_1^4 + \dots - 2N_2^2N_3^2 \dots = -3L^4 - 6L^2M^2\wp v + 9M^4\wp^2 v - 2M^4\wp'' v; \tag{70}$$

so that

$$a + D = a\frac{(L^2+3M^2\wp v)^2 - 4aL\surd(-\Sigma) - 2M^4\wp'' v}{M^4}, \tag{71}$$

an expression analogous to that in equation (214), p. 581, Proc. London Math. Society, vol. XXVII.

We may also write

$$E = -2\frac{Mi\wp' v}{L}, \tag{72}$$

and then by analogy,

$$D = 2\frac{Mi\wp' w}{B}, \tag{73}$$

where

$$w = v_1 - v_2, \tag{74}$$

again a different use of w.

11. Another resolution of the quartic Z can be given, by means of the elliptic functions of the argument v_3, which corresponds to the *infinite* value of z; namely,

$$\surd(a)(z - z_0) = \frac{-\wp' v_3}{\wp u - \wp v_3}, \tag{75}$$

$$\surd a(z - z_a) = \frac{-\wp' v_3}{\wp u - \wp v_3}\frac{\wp u - e_a}{\wp v_3 - e_a}. \tag{76}$$

Differentiating,

$$\begin{aligned}\surd(aZ) = \surd a\frac{dz}{du} &= \frac{\wp' u\,\wp' v_3}{(\wp u - \wp v_3)^2}\\ &= \wp(u - v_3) - \wp(u + v_3),\end{aligned} \tag{77}$$

and integrating,

$$\begin{aligned}\surd(a)\,z &= \zeta(u + v_3) - \zeta(u - v_3) - \zeta 2v_3\\ &= \tfrac{1}{2}\frac{\wp'(u - v_3) - \wp' 2v_3}{\wp(u - v_3) - \wp 2v_3}\\ &= \tfrac{1}{2}\frac{\wp'(u - v_3) + \wp'(u + v_3)}{\wp(u - v_3) - \wp(u + v_3)};\end{aligned} \tag{78}$$

and squaring,

$$az^2 = \wp(u - v_3) + \wp(u + v_3) + \wp 2v_3. \tag{79}$$

12. Putting $u_1 = u_2 = u$ in formula (E) leads to Hermite transformation

$$\wp 2u = -\frac{H}{Z}, \tag{80}$$

where H is the Hessian of the quartic Z, namely,

$$H = (ac - b^2)z^4 + 2(ad - bc)z^3 + (ae + 2bd - 3c^2)z^2 + 2(be - cd)z + ce - d^2. \tag{81}$$

Since $b = 0$ in our form of the quartic Z (20), then corresponding to $z = \infty$,

$$\wp 2v_3 = -c, \tag{82}$$

or $6\wp 2v_3$ is the coefficient of $-z^2$ in Z; so that

$$\begin{aligned} \wp 2v_3 &= \tfrac{1}{3}a + \tfrac{1}{6}D + \tfrac{2}{3}\frac{B^2}{M^2} \\ &= \tfrac{1}{3}a + \tfrac{1}{6}E + \tfrac{2}{3}\frac{L^2}{M^2} \\ &= a\tfrac{1}{3}\frac{N_1^2 + N_2^2 + N_3^2}{M^2} \\ &= \frac{L^2}{M^2} + \wp(v_1 + v_2), \end{aligned} \tag{83}$$

or

$$\wp(v_1 + v_2) = \wp 2v_3 - \frac{L^2}{M^2}; \tag{84}$$

and, by analogy,

$$\wp(v_1 - v_2) = \wp 2v_3 - \frac{B^2}{M^2}; \tag{85}$$

or

$$\wp(v_1 + v_2) = \tfrac{1}{3}a + \tfrac{1}{6}E - \tfrac{1}{3}\frac{L^2}{M^2}, \tag{86}$$

$$\wp(v_1 - v_2) = \tfrac{1}{3}a + \tfrac{1}{6}D - \frac{B^2}{M^2}; \tag{87}$$

while from (72) and (73),

$$i\wp'(v_1 + v_2) = -\frac{LE}{2M},$$

$$i\wp'(v_1 - v_2) = \frac{BD}{2M} \tag{90}$$

So also Hermite's formula

$$\wp' 2u = \frac{G}{Z^{\frac{3}{2}}}, \tag{91}$$

where G denotes the sextic covariant of Z, leads for the infinite value of z to

$$\frac{\wp' 2v_3}{\sqrt{a}} = \text{coefficient of } 4z \text{ in } Z,$$

or

$$\wp' 2v_3 = 2\sqrt{a}\,\frac{BL}{M^2}. \tag{92}$$

We notice that v_3 is a fraction of the real or the imaginary period, according as a is $+1$ or -1, or as the body is prolate or oblate; and, sometimes, by retaining a, we are able to state the results in a general form, suitable for all cases.

13. The annexed diagrams are intended to illustrate the various cases which may arise, and to exhibit to the eye the separation of the roots of the quartic Z; the curves on the left hand are the typical graphs of the function Z, while the concentric circles on the right show the correspondence of z on the outer circle with the elliptic argument u on the inner circle, the shaded portion representing the limits of the actual variation of z.

Case I applies to the prolate body for which $a = +1$; and all four roots of Z are real, and arranged in the order

$$\infty > z_0 > 1 > z_3 > z > z_2 > -1 > z_1 > -\infty,$$

so that ω_1, ω_3 denoting the real and imaginary periods of $\wp u$, and the letter f being employed to denote generically any fraction,

$$v_1 = f\omega_3,\quad v_2 = \omega_1 + f\omega_3,\quad v_3 = f\omega_1,$$
$$v \text{ or } w = v_1 \pm v_2 = \omega_1 + f\omega_3,$$
$$u = \omega_3 + f\omega_1.$$

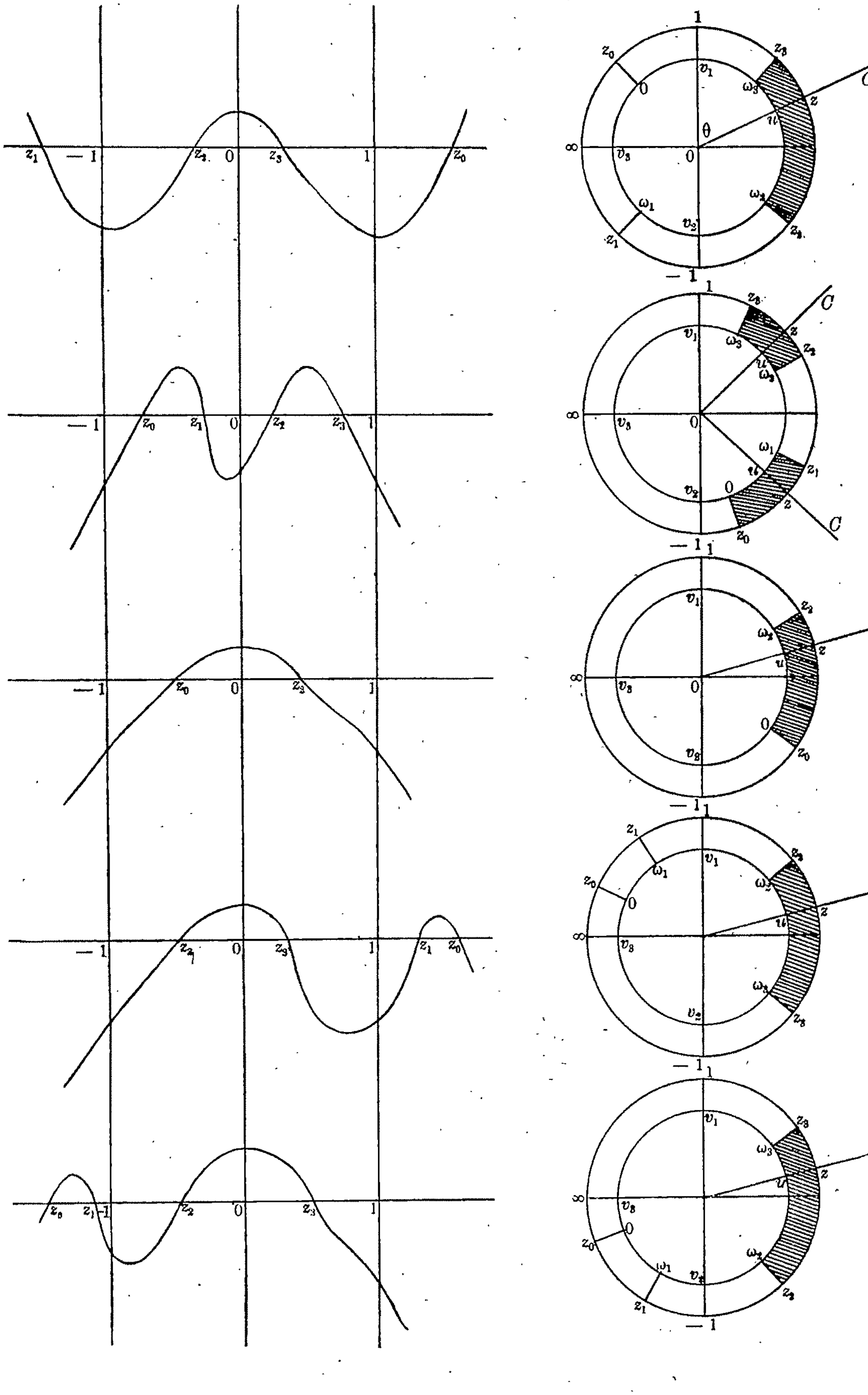
Case II, an oblate body, $a = -1$; all four roots of Z real and arranged in the order

$$\infty > 1 > z_3 > z > z_2 > z_1 > z > z_0 > -1 > -\infty,$$

$$v_1,\ v_2,\ v_3,\ v,\ w = f\omega_3,$$
$$u = \omega_3 + f\omega_1 \text{ or } f\omega_1.$$

Case III, an oblate body, $a = -1$; two real roots of Z and two imaginary:

$$\infty > 1 > z_0 > z > z_2 > -1 > -\infty,$$
$$v_1,\ v_2,\ v_3,\ v,\ w = f\omega_2',$$
$$u = f\omega_2.$$



I

II

III

IV

V

Cases IV and V represent the conditions for an oblate body when the kinetic energy assumes the more general form discussed by Halphen and others, in consequence of which Z takes the more general form, with $a = -1$,

$$Z = -(z^2 - 1)(z^2 - 4bz - 1 + D) - 4\left(\frac{Bx - L}{M}\right)^2$$

or

$$= -(z^2 - 1)(z^2 - 4bz - 1 + E) - 4\left(\frac{Lx - B}{M}\right)^2; \tag{93}$$

and now in

Case IV, $\infty > z_0 > z_1 > 1 > z_2 > z > z_3 > -1 > -\infty,$
$v_1 = \omega_1 + f\omega_3,$ v_2 and $v_3 = f\omega_3,$
v and $w = \omega_1 + f\omega_3.$

Case V. $\infty > 1 > z_3 > z > z_2 > -1 > z_1 > z_0 > -\infty,$
v_1 and $v_3 = f\omega_3,$ $v_2 = \omega_1 + f\omega_3,$
v and $w = \omega_1 + f\omega_3.$

The term b may arise from the external shape or from circulation in the liquid when the body is perforated or ring-shaped (Basset, Hydrodynamics, I, §193), but the absence of b simplifies considerably the elliptic-function expressions.

14. Putting $z = \pm 1$ in (77) and (79),

$$2\sqrt{(-a)}\,\frac{B - L}{M} = \frac{\wp' v_1 \wp' v_3}{(\wp v_1 - \wp v_3)^2} = \wp(v_1 - v_3) - \wp(v_1 + v_3), \tag{94}$$

$$2\sqrt{(-a)}\,\frac{B + L}{M} = \frac{\wp' v_2 \wp' v_3}{(\wp v_2 - \wp v_3)^2} = \wp(v_2 - v_3) - \wp(v_2 + v_3); \tag{95}$$

$$a = \wp(v_1 - v_3) + \wp(v_1 + v_3) + \wp 2v_3, \tag{96}$$

$$a = \wp(v_2 - v_3) + \wp(v_2 + v_3) + \wp 2v_3. \tag{97}$$

Thence

$$\wp(v_1 - v_3) = \tfrac{1}{2}a - \tfrac{1}{2}\wp 2v_3 + \sqrt{(-a)}\,\frac{B - L}{M}, \tag{98}$$

$$\wp(v_1 + v_3) = \tfrac{1}{2}a - \tfrac{1}{2}\wp 2v_3 - \sqrt{(-a)}\,\frac{B - L}{M}, \tag{99}$$

$$\wp(v_2 - v_3) = \tfrac{1}{2}a - \tfrac{1}{2}\wp 2v_3 + \sqrt{(-a)}\,\frac{B + L}{M}, \tag{100}$$

$$\wp(v_2 + v_3) = \tfrac{1}{2}a - \tfrac{1}{2}\wp 2v_3 - \sqrt{(-a)}\,\frac{B + L}{M}. \tag{101}$$

Again, putting $z = \pm 1$ in (78),

$$\wp'(v_1 - v_3) - \wp' 2v_3 = 2\sqrt{a}\,\{\wp(v_1 - v_3) - \wp 2v_3\}, \tag{102}$$

$$\wp'(v_1 - v_3) = 2\sqrt{a}\left\{\frac{BL}{M^2} + \tfrac{1}{3}a - \tfrac{3}{2}\wp 2v_3 + \sqrt{(-a)}\,\frac{B-L}{M}\right\}$$

$$= 2\sqrt{a}\,\frac{BL}{M^2} - \tfrac{1}{2}\sqrt{a}E - 2\sqrt{a}\,\frac{L^2}{M^2} + 2i\,\frac{B-L}{M}, \tag{103}$$

and

$$\wp'(v_1 - v_3) + \wp'(v_1 + v_3) = 2\sqrt{a}\,\{\wp(v_1 - v_3) - \wp(v_1 + v_3)\}$$

$$= 4i\,\frac{B-L}{M}, \tag{104}$$

$$\wp'(v_1 + v_3) = -2\sqrt{a}\,\frac{BL}{M^2} + \tfrac{1}{2}\sqrt{a}E + 2\sqrt{a}\,\frac{L^2}{M^2} + 2i\,\frac{B-L}{M}. \tag{105}$$

Similarly

$$\wp'(v_2 - v_3) = 2\sqrt{a}\,\frac{BL}{M^2} + \tfrac{1}{2}\sqrt{a}E + 2\sqrt{a}\,\frac{L^2}{M^2} - 2i\,\frac{B+L}{M}, \tag{106}$$

$$\wp'(v_2 + v_3) = -2\sqrt{a}\,\frac{BL}{M^2} - \tfrac{1}{2}\sqrt{a}E - 2\sqrt{a}\,\frac{L^2}{M^2} - 2i\,\frac{B+L}{M}. \tag{107}$$

Thus

$$\wp(v_1 - v_3) - \wp(v_2 + v_3) = 2\sqrt{(-a)}\,\frac{BL}{M}, \tag{108}$$

$$\wp'(v_1 - v_3) - \wp'(v_2 + v_3) = 4\sqrt{a}\,\frac{BL}{M^2} + 4i\,\frac{B}{M}; \tag{109}$$

so that

$$\tfrac{1}{2}\,\frac{\wp'(v_1 - v_3) - \wp'(v_2 + v_3)}{\wp(v_1 - v_3) - \wp(v_2 + v_3)} = -i\,\frac{L}{M} + \sqrt{a}; \tag{110}$$

and squaring,

$$\wp(v_1 + v_2) + \wp(v_1 - v_3) + \wp(v_2 + v_3) = -\frac{L^2}{M^2} - 2\sqrt{(-a)}\,\frac{L}{M} + a. \tag{111}$$

But
$$\wp(v_1 - v_3) + \wp(v_2 + v_3) = a - \wp 2v_3 - 2\sqrt{(-a)}\,\frac{L}{M}; \tag{112}$$

so that
$$\wp(v_1 + v_2) = \wp 2v_3 - \frac{L^2}{M^2}, \tag{113}$$

agreeing with equation (84).

Similarly,

$$\wp(v_1+v_3)-\wp(v_2+v_3)= \quad 2\surd(-a)\frac{L}{M},$$

$$\wp'(v_1+v_3)+\wp'(v_2+v_3)=-4\surd a\frac{BL}{M^2}-4i\frac{L}{M},$$

$$\tfrac{1}{2}\frac{\wp'(v_1+v_3)+\wp'(v_2+v_3)}{\wp(v_1+v_3)-\wp(v_2+v_3)}=+i\frac{B}{M}-\surd a; \tag{114}$$

so that, squaring,

$$\wp(v_1-v_2)+\wp(v_1+v_3)+\wp(v_2+v_3)=-\frac{B^2}{M^2}-2\surd(-a)\frac{B}{M}+a. \tag{115}$$

But

$$\wp(v_1+v_3)+\wp(v_2+v_3)=a-\wp 2v_3-2\surd(-a)\frac{B}{M}, \tag{116}$$

so that

$$\wp(v_1-v_2)=\wp 2v_3-\frac{B^2}{M^2}, \tag{117}$$

agreeing with equation (85).

15. Again, from (110),

$$\tfrac{1}{2}\frac{-\wp'(v_1+v_2)-\wp'(v_1-v_3)}{\wp(v_1+v_2)-\wp(v_1-v_3)}=-i\frac{L}{M}+\surd a, \tag{118}$$

and

$$\begin{aligned}\wp(v_1+v_2)-\wp(v_1-v_3)&=\wp 2v_3-\frac{L^2}{M^2}-\tfrac{1}{2}a+\tfrac{1}{2}\wp 2v_3-\surd(-a)\frac{B-L}{M}\\ &=\tfrac{1}{4}E-\surd(-a)\frac{B-L}{M},\end{aligned} \tag{119}$$

therefore

$$\begin{aligned}\wp'(v_1+v_2)&=2\left(i\frac{L}{M}-\surd a\right)\{\wp(v_1+v_2)-\wp(v_2-v_3)\}-\wp'(v_2-v_3)\\ &=\left(i\frac{L}{M}-\surd a\right)\left\{\tfrac{1}{2}E-2\surd(-a)\frac{B-L}{M}\right\}\\ &\quad-2\surd a\frac{BL}{M^2}+\tfrac{1}{2}\surd aE+2\surd a\frac{L^2}{M^2}-2i\frac{B-L}{M}\\ &=\tfrac{1}{2}i\frac{LE}{M},\end{aligned} \tag{120}$$

as before in (89).

So also, from (114),

$$\begin{aligned}\tfrac{1}{2}\frac{\wp'(v_1-v_2)-\wp'(v_2+v_3)}{\wp(v_1-v_2)-\wp(v_2+v_3)}&=\tfrac{1}{2}\frac{-\wp'(v_1+v_3)-\wp'(v_2+v_3)}{\wp(v_1+v_3)-\wp(v_2+v_3)}\\ &=-i\frac{B}{M}+\surd a,\end{aligned} \tag{121}$$

and

$$\wp(v_1 - v_2) - \wp(v_2 + v_3) = \wp 2v_3 - \frac{B^2}{M^2} - \tfrac{1}{2}a + \tfrac{1}{2}\wp 2v_3 + \sqrt{(-a)}\,\frac{B+L}{M}$$

$$= \tfrac{1}{4}D + \sqrt{(-a)}\,\frac{B+L}{M}; \tag{122}$$

therefore

$$\wp'(v_1 - v_2) = \left(-i\frac{B}{M} + \sqrt{a}\right)\left\{\tfrac{1}{2}D + 2\sqrt{(-a)}\,\frac{B+L}{M}\right\}$$

$$-2\sqrt{a}\,\frac{BL}{M^2} - \tfrac{1}{2}\sqrt{a}D - 2\sqrt{a}\,\frac{B^2}{M^2} - 2i\,\frac{B+L}{M} = -\tfrac{1}{2}i\,\frac{BD}{M}, \tag{123}$$

agreeing with (90).

These verifications are useful in fixing the signs of the various expressions, in the ambiguous cases, of frequent occurrence in these calculations.

16. Returning to (75) and putting in it $u = v_1$, and $z = 1$,

$$\sqrt{a}\,(1 - z_0) = \frac{-\wp' v_3}{\wp v_1 - \wp v_3}, \tag{124}$$

so that

$$\sqrt{a}\,(1 - z) = \frac{-\wp' v_3\,(\wp u - \wp v_1)}{(\wp v_1 - \wp v_3)(\wp u - \wp v_3)}; \tag{125}$$

and similarly,

$$\sqrt{a}\,(-1 - z) = \frac{-\wp' v_3\,(\wp u - v_2)}{(\wp v_2 - \wp v_3)(\wp u - \wp v_3)}. \tag{126}$$

Also, from (24), (25), (44), (45) and (77),

$$\frac{G - CrF}{2AnF} = \frac{L - B}{M} = -\frac{1}{2\sqrt{(-a)}}\,\frac{\wp' v_1\,\wp' v_3}{(\wp v_1 - \wp v_3)^2}, \tag{127}$$

$$\frac{G + CrF}{2AnF} = \frac{L + B}{M} = \frac{1}{2\sqrt{(-a)}}\,\frac{\wp' v_2\,\wp' v_3}{(\wp v_2 - \wp v_3)^2}; \tag{128}$$

so that in (14)

$$\frac{d\psi_1 i}{du} = i\,\frac{G - CrF}{2AnF}\,\frac{1}{1 - z}$$

$$= \tfrac{1}{2}\,\frac{\wp' v_1\,(\wp u - \wp v_3)}{(\wp v_1 - \wp v_3)(\wp u - \wp v_1)}$$

$$= \tfrac{1}{2}\,\frac{\wp' v_1}{\wp v_1 - \wp v_3} + \tfrac{1}{2}\,\frac{\wp' v_1}{\wp u - \wp v_1}$$

$$= \tfrac{1}{2}\zeta(v_1 + v_3) + \tfrac{1}{2}\zeta(v_1 - v_3) - \zeta v_1$$

$$- \tfrac{1}{2}\zeta(u - v_1) + \tfrac{1}{2}\zeta(u + v_1) + \zeta v_1. \tag{129}$$

Similarly

$$\frac{d\psi_2 i}{du} = i\frac{G + CrF}{2AnF}\frac{1}{1+z}$$

$$= \tfrac{1}{2}\frac{\wp' v_2(\wp u - \wp v_3)}{(\wp v_2 - \wp v_3)(\wp u - \wp v_2)}$$

$$= \tfrac{1}{2}\frac{\wp' v_2}{\wp v_2 - \wp v_3} + \tfrac{1}{2}\frac{\wp' v_2}{\wp u - \wp v_2}$$

$$= \tfrac{1}{2}\zeta(v_2 + v_3) + \tfrac{1}{2}\zeta(v_2 - v_3) - \zeta v_2$$

$$- \tfrac{1}{2}\zeta(u - v_2) + \tfrac{1}{2}\zeta(u + v_2) + \zeta v_2. \tag{130}$$

Integrating (129) and (130),

$$\psi_1 i = \tfrac{1}{2}\{\zeta(v_1 + v_3) + \zeta(v_1 - v_3)\} nt + \tfrac{1}{2}\log\frac{\sigma(u+v_1)}{\sigma(u-v_1)}, \tag{131}$$

$$\psi_2 i = \tfrac{1}{2}\{\zeta(v_2 + v_3) + \zeta(v_2 - v_3)\} nt + \tfrac{1}{2}\log\frac{\sigma(u+v_2)}{\sigma(u-v_2)}; \tag{132}$$

and adding,

$$\psi i = \tfrac{1}{2} Qnt + \tfrac{1}{2}\log\frac{\sigma(u+v_1)\,\sigma(u+v_2)}{\sigma(u-v_1)\,\sigma(u-v_2)}, \tag{133}$$

where $$Q = \zeta(v_1 + v_3) + \zeta(v_1 - v_3) + \zeta(v_2 + v_3) + \zeta(v_2 - v_3). \tag{134}$$

By a theorem of Elliptic Functions,

$$\tfrac{1}{2}\log\frac{\sigma(u+v_1)\,\sigma(u+v_2)}{\sigma(u-v_1)\,\sigma(u-v_2)} = \tfrac{1}{2}\log\frac{\sigma(u+v)}{\sigma(u-v)} + \xi i, \tag{135}$$

where $$v_1 + v_2 = v, \tag{51}$$

and

$$\xi = \tan^{-1}\frac{(\wp v_1 - \wp v_2)\,\wp' u}{i\wp' v_2(\wp u - \wp v_1) - i\wp' v_1(\wp u - \wp v_2)}$$

$$= \sin^{-1}\frac{\wp' u}{\sqrt{\{(\wp u - \wp v_1)(\wp u - \wp v_2)(\wp u - \wp v)\}}}, \tag{136}$$

so that we can put

$$\psi i = \tfrac{1}{2} Qnt + \xi i + \tfrac{1}{2}\log\frac{\sigma(u+v)}{\sigma(u-v)}, \tag{137}$$

and we have now practically added the parameters v_1 and v_2 of two Elliptic Integrals of the third kind into a single parameter v.

17. We now introduce the Elliptic Integral of the third kind, with elliptic parameter v, in the standard form we shall employ and denote by $I(v)$, namely,

$$I(v) = \tfrac{1}{2}\int \frac{P(v)(s-\sigma) - \sqrt{-\Sigma}}{(s-\sigma)\sqrt{S}}$$

$$= \tfrac{1}{2} i \log \frac{\sigma(u-v)}{\sigma(u+v)} e^{\left(2\zeta v - i\frac{P}{M}\right)u \text{ (or } nt)}, \tag{138}$$

with

$$\int \frac{ds}{\sqrt{S}} = \frac{u}{M} \quad \text{or} \quad \frac{nt}{M}, \tag{139}$$

where $P(v)$ or P is a certain function of v, to be defined hereafter, so chosen as to cancel the secular term when the integral $I(v)$ is *pseudo-elliptic*, in consequence of the parameter v being an aliquot part of a period.

Now from (138),

$$\tfrac{1}{2} \log \frac{\sigma(u+v)}{\sigma(u-v)} = iI(v) - nt\zeta v - \tfrac{1}{2} i \frac{P}{M} nt, \tag{140}$$

and thus

$$\psi i = \tfrac{1}{2}\Big\{\zeta(v_1+v_3) + \zeta(v_2-v_3) - \zeta(v_1+v_2)$$
$$+ \zeta(v_2+v_3) + \zeta(v_1-v_3) - \zeta(v_1+v_2) - i\frac{P}{M}\Big\} nt + iI(v) + \xi i. \tag{141}$$

But, from (99) and (100),

$$\zeta(v_1+v_3) + \zeta(v_2-v_3) - \zeta(v_1+v_2)$$
$$= \sqrt{\{\wp(v_1+v_3) + \wp(v_2-v_3) + \wp(v_1+v_2)\}}$$
$$= \sqrt{\Big\{\tfrac{1}{2}a - \tfrac{1}{2}\wp 2v_3 - \sqrt{(-a)}\frac{B-L}{M}}$$
$$\quad + \tfrac{1}{2}a - \tfrac{1}{2}\wp 2v_3 + \sqrt{(-a)}\frac{B+L}{M} + \wp 2v_3 - \frac{L^2}{M^2}\Big\}$$
$$= \sqrt{\left\{a + 2\sqrt{(-a)}\frac{L}{M} - \frac{L^2}{M^2}\right\}} \quad = i\frac{L}{M} + \sqrt{a}, \tag{142}$$

and similarly,

$$\zeta(v_2+v_3) + \zeta(v_1-v_3) - \zeta(v_1+v_2) = i\frac{L}{M} - \sqrt{a}. \tag{143}$$

To settle the ambiguity of sign, we may also employ the formulas

$$\zeta(v_1+v_3) + \zeta(v_2-v_3) - \zeta(v_1+v_2) = -\tfrac{1}{2}\frac{\wp'(v_1+v_3) - \wp'(v_2-v_3)}{\wp(v_1+v_3) - \wp(v_2-v_3)},$$
$$\zeta(v_2+v_3) + \zeta(v_1-v_3) - \zeta(v_1+v_2) = -\tfrac{1}{2}\frac{\wp'(v_2+v_3) - \wp'(v_1-v_3)}{\wp(v_2+v_3) - \wp(v_1-v_3)}, \tag{144}$$

and thus

$$\psi i = \left(i\frac{L}{M} - \tfrac{1}{2}i\frac{P}{M}\right)nt + iI(v) + \xi i,$$

and, dropping the factor i,

$$\psi = \frac{L - \frac{1}{2}P(v)}{M}nt + I(v) + \xi, \tag{145}$$

or

$$\psi - pt = I(v) + \xi, \tag{146}$$

where

$$\frac{p}{n} = \frac{L - \frac{1}{2}P(v)}{M}. \tag{147}$$

a different p to the component angular velocity about OA in §1. Without this preliminary determination of pt, the secular term in ψ, the consideration of the pseudo-elliptic solutions would be hopeless.

18. The next chief object of the present paper is the discussion of the Pseudo-Elliptic cases which arise when the parameter v is made an aliquot part, one n^{th}, of a period, so as to be of the form

$$v = \omega_1 + \frac{2\omega_3}{n}, \text{ or } \frac{2\omega_3}{n}, \tag{148}$$

according as the body is prolate or oblate; and when $P(v)$ is at the same time so chosen as to make $I(v)$ an inverse circular function; the preliminary analysis will be found in the paper on "Pseudo-Elliptic Integrals," Proc. London Math. Society, vol. XXV, from which the results required in the sequel will be taken.

In such cases it will be found that we must take

$$\frac{iP(v)}{M} = \zeta v - \frac{\eta v}{\omega}; \tag{149}$$

and

$$iI(v) = \tfrac{1}{2}\log\frac{\sigma(u+v)}{\sigma(u-v)}e^{su}, \tag{150}$$

where

$$v = \frac{2\varpi}{n}, \quad s = \frac{2\eta}{n}; \tag{151}$$

and now $iI(v)$ is the logarithm of a function, analogous to the sn, cn or dn function, which is considered in Halphen's "Fonctions elliptiques," I, p. 224, the function being the n^{th} root of a rational function of $\wp u$ and $\wp' u$.

Further, by taking

$$L = \tfrac{1}{2}P(v), \tag{152}$$

the secular term pt is cancelled in the equations and all trace of elliptic transcendentalism is eliminated, so that the various curves described by points in the axis of the body are, relatively to the moving origin O, purely algebraical curves; these special cases are interesting to discuss and to represent diagrammatically and stereoscopically, like the analogous algebraical Spherical Catenaries and Gyrostat Curves, investigated in the Proc. London Math. Society, vol. XXVII, and drawn by Mr. T. I. Dewar.

19. The curve described by the projection of the moving origin O on a plane perpendicular to Oz is intimately associated with the cone described by the axis of figure OC round Oz; for, denoting the coordinates of the projection of O by α, β, and the advance of O parallel to Oz by γ, then, according to the equations given in Kirchhoff's "Vorlesungen," p. 240,

$$F\alpha = \beta_1 \frac{\partial T}{\partial p} + \beta_2 \frac{\partial T}{\partial q} + \beta_3 \frac{\partial T}{\partial r}, \tag{153}$$

$$F\beta = -\alpha_1 \frac{\partial T}{\partial p} - \alpha_2 \frac{\partial T}{\partial q} - \alpha_3 \frac{\partial T}{\partial r}, \tag{154}$$

$$F\frac{d\gamma}{dt} = u\frac{\partial T}{\partial u} + v\frac{\partial T}{\partial v} + w\frac{\partial T}{\partial w}; \tag{155}$$

where u, v, w again denote the component velocities of O in the directions OA, OB, OC; and α_1, α_2, α_3 denote the cosines of the angles between $O'\alpha$ and OA, OB, OC; and β_1, β_2, β_3 the cosines of the angles between $O'\beta$ and OA, OB, OC; $O'\alpha$, $O'\beta$, $O'\gamma$ being three fixed axes in space, drawn through a fixed origin O', $O'\gamma$ being drawn parallel to Oz.

Expressed by means of Euler's angles θ, ϕ, ψ,

$$\begin{aligned} \alpha_1 &= \cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi, \\ \alpha_2 &= -\cos\theta\sin\phi\cos\psi - \cos\phi\sin\psi, \\ \alpha_3 &= \sin\theta\cos\psi; \end{aligned} \tag{156}$$

$$\begin{aligned} \beta_1 &= \cos\theta\cos\phi\sin\psi + \sin\phi\cos\psi, \\ \beta_2 &= -\cos\theta\sin\phi\sin\psi + \cos\phi\cos\psi, \\ \beta_3 &= \sin\theta\sin\psi; \end{aligned} \tag{157}$$

while

$$\begin{aligned} p &= \sin\phi\dot\theta - \sin\theta\cos\phi\dot\psi, \\ q &= \cos\phi\dot\theta + \sin\theta\sin\phi\dot\psi, \\ r &= \dot\phi + \cos\theta\dot\psi; \end{aligned} \tag{158}$$

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so that, after reduction,

$$F\alpha = A \cos\psi\dot{\theta} + (Cr - A\cos\theta\dot{\psi})\sin\theta\sin\psi.$$

$$F\beta = A\sin\psi\dot{\theta} - (Cr - A\cos\theta\dot{\psi})\sin\theta\cos\psi, \tag{159}$$

$$F\frac{d\alpha}{dt} = aAn^2\sin\theta\cos\theta\cos\psi,$$

$$F\frac{d\beta}{dt} = aAn^2\sin\theta\cos\theta\sin\psi,$$

$$F\frac{d\gamma}{dt} = \frac{F^2}{P} + aAn^2\cos^2\theta. \tag{160}$$

20. Changing to polar coordinates ρ, ϖ in the plane $O'\alpha\beta$, so that

$$\alpha = \rho\cos\varpi,\quad \beta = \rho\sin\varpi, \tag{161}$$

then

$$F(\alpha + \beta i) = F\rho e^{\varpi i} = \{A\dot{\theta} - i\sin\theta(Cr - A\cos\theta\dot{\psi})\}e^{\psi i}, \tag{162}$$

and

$$F\rho\cos(\psi - \varpi) = A\dot{\theta} = -\frac{An\sqrt{Z}}{\sin\theta}, \tag{163}$$

$$F\rho\sin(\psi - \varpi) = (Cr - A\cos\theta\dot{\psi})\sin\theta$$
$$= \frac{CrF - G\cos\theta}{F\sin\theta}. \tag{164}$$

Thus, squaring and adding,

$$F^2\rho^2 = \frac{A^2n^2F^2Z + (CrF - Gz)^2}{F^2(1-z^2)}$$
$$= A^2n^2a(aE + 1 - z^2); \tag{165}$$

and, dividing,

$$\tan(\psi - \varpi) = -\frac{CrF - Gz}{AnF\sqrt{Z}}, \tag{166}$$

$$\varpi = \psi + \tan^{-1}\frac{CrF - Gz}{AnF\sqrt{Z}}$$
$$= \psi + \sin^{-1}\frac{CrF - G\cos\theta}{F^2\rho\sin\theta}$$
$$= \psi + \cos^{-1}\frac{\sqrt{Z}}{\sqrt{\{a(1-z^2)(aE+1-z^2)\}}}; \tag{167}$$

so that, from equation (146),

$$\varpi = pt + I(v) + \xi + \sin^{-1}\frac{CrF - G\cos\theta}{F^2\rho\sin\theta}, \tag{168}$$

and ϖ and ψ depend upon the same elliptic integral $I(v)$.

We find in fact, by differentiation of (167),

$$\frac{d\varpi}{dz} = \frac{d\psi}{dz} + \frac{d}{dz}\sin^{-1}\frac{CrF - Gz}{AnF\surd\{a(1-z^2)(aE+1-z^2)\}}$$

$$= \frac{G - CrFz}{AnF(1-z^2)}\frac{1}{\surd Z}$$

$$+ \frac{\dfrac{CrF - Gz}{AnF\surd\{a(1-z^2)(aE+1-z^2)\}}\left(\dfrac{-G}{CrF-Gz} + \dfrac{z}{1-z^2} + \dfrac{z}{aE+1-z^2}\right)}{\dfrac{\surd Z}{\surd\{a(1-z^2)(aE+1-z^2)\}}}$$

$$= \left\{\frac{G - CrFz}{AnF(1-z^2)} - \frac{G}{AnF} + \frac{CrFz - Gz^2}{AnF(1-z^2)} + \frac{CrFz - Gz^2}{AnF(aE+1-z^2)}\right\}\frac{1}{\surd Z}$$

$$= \frac{CrFz - Gz^2}{AnF(aE+1-z^2)}\frac{1}{\surd Z}$$

$$= 2\frac{Bz - Lz^2}{M(aE+1-z^2)}\frac{1}{\surd Z}. \tag{169}$$

Now if w_1, w_2 denote the values of the elliptic argument u corresponding to

$$aE + 1 - z^2 = 0, \tag{170}$$

it follows by Abel's theorem and the theory of elliptic functions that

$$w_1 + w_2 = v_1 + v_2 = v. \tag{171}$$

So also if t_1, t_2 denote the values of u corresponding to

$$aD + 1 - z^2 = 0, \tag{172}$$

then

$$t_1 - t_2 = v_1 - v_2 = w. \tag{173}$$

21. From equations (125) and (126),

$$\sin^2\theta = 1 - z^2 = -a\frac{\wp'^2 v_3(\wp u - \wp v_1)(\wp u - \wp v_2)}{(\wp v_1 - \wp v_3)(\wp v_2 - \wp v_3)(\wp u - \wp v_3)^2}$$

$$= C\frac{\sigma(u - v_1)\sigma(u + v_1)\sigma(u - v_2)\sigma(u + v_2)}{\sigma^2(u - v_3)\sigma^2(u + v_3)}, \tag{174}$$

and from (133),

$$e^{2\psi i} = \frac{\sigma(u + v_1)\sigma(u + v_2)}{\sigma(u - v_1)\sigma(u - v_2)}e^{Qnt}; \tag{175}$$

so that by multiplication,

$$\sin\theta e^{\psi i} = \surd C\frac{\sigma(u + v_1)\sigma(u + v_2)}{\sigma(u - v_3)\sigma(u + v_3)}e^{\frac{1}{2}Qnt}. \tag{176}$$

Or, in Klein's manner, with the stereographic projection,

$$\tan \tfrac{1}{2}\theta e^{\psi i} = C' \frac{\sigma(u+v_1)\,\sigma(u+v_2)}{\sigma(u-v_1)\,\sigma(u-v_2)} e^{\frac{1}{2}Qnt}. \tag{177}$$

Similarly we find

$$\rho e^{\varpi i} = C'' \frac{\sigma(u+w_1)\,\sigma(u+w_2)}{\sigma(u-v_3)\,\sigma(u+v_3)} e^{\frac{1}{2}Qnt}, \tag{178}$$

so that the expressions in Halphen's equations (52), F. E. II, p. 164, appear to require correction; however, the curve of (α, β) is intimately connected with the cone of (θ, ψ), and the two are pseudo-elliptic, and even algebraical, at the same time.

As for u, v and p, q, they depend upon the same elliptic functions as Euler's angle ϕ; for

$$P(u+vi) = -F\sin\theta\, e^{-\phi i}, \tag{179}$$

$$p + qi = (-\sin\theta\dot{\psi} + i\dot{\theta})\, e^{-\phi i}$$

$$= -n \frac{2\dfrac{L-Bz}{M} + i\sqrt{Z}}{\sqrt{(1-z^2)}} e^{-\phi i}, \tag{180}$$

which are pseudo-elliptic together with ϕ; and this is the case when

$$v_1 - v_2 = t_1 - t_2 = w \tag{181}$$

is an aliquot part of a period; but these cases are not so interesting from the dynamical point of view.

22. In the pseudo-elliptic case of the motion of a prolate solid, when the parameter

$$v = \omega_1 + \frac{2\omega_3}{n}, \tag{182}$$

where n is an odd integer, the expressions for ψ and ϖ must be of the form

$$\psi - pt = \frac{1}{n}\cos^{-1}\frac{Hz^{n-1} + H_1 z^{n-2} + \ldots + H_{n-1}}{(1-z^2)^{\frac{1}{2}n}}\sqrt{(z_0 - z \,.\, z - z_1)}$$

$$= \frac{1}{n}\sin^{-1}\frac{Kz^{n-1} + K_1 z^{n-2} + \ldots + K_{n-1}}{(1-z^2)^{\frac{1}{2}n}}\sqrt{(z_3 - z \,.\, z - z_2)}, \tag{183}$$

and

$$\varpi - pt = \frac{1}{n}\cos^{-1}\frac{Iz^{n-1} + I_1 z^{n-2} + \ldots + I_{n-1}}{(E+1-z^2)^{\frac{1}{2}n}}\sqrt{(z_0 - z \,.\, z - z_1)}$$

$$= \frac{1}{n}\sin^{-1}\frac{Jz^{n-1} + J_1 z^{n-2} + \ldots + J_{n-1}}{(E+1-z^2)^{\frac{1}{2}n}}\sqrt{(z_3 - z \,.\, z - z_2)}. \tag{184}$$

But with a parameter

$$v = \omega_1 + \frac{\omega_2}{n}, \tag{185}$$

where n is an integer, the z_1 and z_2 or z_3 must change places, as well as the $\cos^{-1}$ and $\sin^{-1}$, taking $\frac{dz}{dt}$ as positive at the start.

With an oblate body the parameter v is a fraction of the imaginary period, and when

$$v = \frac{\omega_3}{n}, \tag{186}$$

where n is an integer, we must have

$$\begin{aligned} \psi - pt &= \frac{1}{n} \cos^{-1} \frac{Hz^{n-1} + H_1 z^{n-2} + \ldots + H_{n-1}}{(1-z^2)^{\frac{1}{2}n}} \sqrt{(z \sim z_1 . z \sim z_2)} \\ &= \frac{1}{n} \sin^{-1} \frac{Kz^{n-1} + K_1 z^{n-2} + \ldots + K_{n-1}}{(1-z^2)^{\frac{1}{2}n}} \sqrt{(z_3 - z . z - z_0)}, \end{aligned} \tag{187}$$

with a similar expression for $\varpi - pt$.

When, in the motion of an oblate body, the parameter

$$v = \frac{2\omega_3}{n}, \tag{188}$$

where n is an odd integer, the quartic Z will not be divided into quadratic factors, and we must have

$$\begin{aligned} \psi - pt &= \frac{1}{n} \cos^{-1} \frac{Hz^n + H_1 z^{n-1} + \ldots + H_n}{(1-z^2)^{\frac{1}{2}n}} \\ &= \frac{1}{n} \sin^{-1} \frac{Kz^{n-2} + K_1 z^{n-3} + \ldots + K_{n-2}}{(1-z^2)^{\frac{1}{2}n}} \sqrt{Z}, \end{aligned} \tag{189}$$

with a similar expression for $\varpi - pt$.

23. Having now chosen a simple numerical value of n, for a case worked out in the paper on "Pseudo-Elliptic Integrals," and having expressed the s_1, s_2, s_3 in (50), or Halphen's x and y in (49), in terms of a single parameter c, as well as σ, $\sqrt{-\Sigma}$ and $P(v)$, and having chosen an arbitrary value of L, the quickest practical procedure for the determination of the coefficients H, K and

4

I, J, appears to be effected from the identical equations obtained by squaring and adding, and also from the differentiation of the two equivalent forms for ψ and ϖ.

Thus in consequence of the relations

$$\frac{dz}{dt} = n\sqrt{Z}, \tag{19}$$

$$\frac{d\psi}{dt} = 2\frac{G - CrFz}{AF(1-z^2)}, \tag{12}$$

we have

$$\frac{d\psi}{dz} = 2\frac{L - Bz}{M(1-z^2)}\frac{1}{\sqrt{Z}}; \tag{46}$$

and putting

$$\psi - pt = \chi, \tag{190}$$

$$\frac{d\chi}{dz} = 2\frac{L - Bz}{M(1-z^2)}\frac{1}{\sqrt{Z}} - \frac{p}{n}\frac{1}{\sqrt{Z}},$$

$$\therefore\ (1-z^2)\sqrt{Z}\frac{d\chi}{dz} = 2\frac{L - Bz}{M} - \frac{p}{n}(1-z^2)$$

$$= \frac{L - \frac{1}{2}P(v)}{M}z^2 - 2\frac{B}{M}z + \frac{L + \frac{1}{2}P(v)}{M}. \tag{191}$$

So also

$$(aE + 1 - z^2)\sqrt{Z}\frac{d(\varpi - pt)}{dz}$$

$$= -\frac{L + \frac{1}{2}P}{M}z^2 + 2\frac{B}{M}z - \frac{L - \frac{1}{2}P}{M}(aE + 1). \tag{192}$$

24. Begin with the simplest pseudo-elliptic case of all, in which

$$v = \omega_a \text{ and } \wp' v = 0; \tag{193}$$

so that, from (72),

$$L = 0, \tag{194}$$

or

$$E = 0. \tag{195}$$

When $L = 0$ or $G = 0$,

$$\frac{d\psi}{dz} = -2\frac{B}{M}\frac{z}{(1-z^2)\sqrt{Z}}, \tag{196}$$

$$\frac{d\varpi}{dz} = 2\frac{B}{M}\frac{z}{(aE + 1 - z^2)\sqrt{Z}}, \tag{197}$$

where Z is a quadratic in z^2, so that taking z^2 as independent variable, ψ and ϖ are non-elliptic, and integrating,

$$\psi = \tfrac{1}{2}\sin^{-1}\frac{2\dfrac{B}{M}\sqrt{Z}}{\Delta(1-z^2)}, \tag{198}$$

$$\varpi = \tfrac{1}{2}\sin^{-1}\frac{2\dfrac{B}{M}\sqrt{Z}}{\Delta(aE+1-z^2)}, \tag{199}$$

where Δ denotes the discriminant of Z.

For a prolate body, $a = +1$,

$$Z = (z_1^2 - z^2)(z_2^2 - z^2), \tag{200}$$

and

$$\begin{aligned} \psi &= \cos^{-1}\sqrt{\frac{(1-z_2^2)(z_1^2-z^2)}{(z_1^2-z_2^2)(1-z^2)}}, \\ &= \sin^{-1}\sqrt{\frac{(z_1^2-1)(z_2^2-z^2)}{(z_1-z_2^2)(1-z^2)}}, \end{aligned} \tag{201}$$

$$\begin{aligned} \varpi &= \sin^{-1}\sqrt{\frac{(E+1-z_2^2)(z_1^2-z^2)}{(z_1^2-z_2^2)(E+1-z^2)}} \\ &= \cos^{-1}\sqrt{\frac{(z_1^2-1-E)(z_2^2-z^2)}{(z_1^2-z_2^2)(E+1-z^2)}}; \end{aligned} \tag{202}$$

which may be written

$$\sqrt{(z_1^2-z_2^2)}\sin\theta e^{\psi i} = \sqrt{(1-z_2^2)}\sqrt{(z_1^2-z^2)} + i\sqrt{(z_1^2-1)}\sqrt{(z_2^2-z^2)}, \tag{203}$$

$$\sqrt{(z_1^2-z_2^2)}\frac{F}{An}\rho e^{\varpi i} = \sqrt{(z_1^2-1-E)}\sqrt{(z_2^2-z^2)} + i\sqrt{(E+1-z_2^2)}\sqrt{(z_1^2-z^2)}, \tag{204}$$

and these equations prove that the cone described by the axis OC round the fixed direction Oz is a quadric cone, while the curve of (α, β) is a conic section, an ellipse.

For an oblate body, $a = -1$; and in Case II,

$$Z = (z_0^2 - z^2)(z^2 - z_1^2) \tag{205}$$

and

$$\begin{aligned} \psi &= \cos^{-1}\sqrt{\frac{(1-z_0^2)(z^2-z_1^2)}{(z_0^2-z_1^2)(1-z^2)}} \\ &= \sin^{-1}\sqrt{\frac{(1-z_1^2)(z_0^2-z^2)}{(z_0^2-z_1^2)(1-z^2)}}, \end{aligned} \tag{206}$$

$$\varpi = \cos^{-1}\sqrt{\frac{(z_0^2 - 1 + E)(z^2 - z_1^2)}{(z_0^2 - z_1^2)(z^2 - 1 + E)}}$$

$$= \sin^{-1}\sqrt{\frac{(z_1^2 - 1 + E)(z_0^2 - z^2)}{(z_0^2 - z_1^2)(z^2 - 1 + E)}}; \tag{207}$$

or

$$\sqrt{(z_0^2 - z_1^2)} \sin\theta e^{\psi i} = \sqrt{(1 - z_0^2)}\sqrt{(z^2 - z_1^2)} + i\sqrt{(1 - z_1^2)}\sqrt{(z_0^2 - z^2)}, \tag{208}$$

$$\sqrt{(z_0^2 - z_1^2)} \frac{F}{An} \rho e^{\varpi i} = \sqrt{(z_0^2 - 1 + E)}\sqrt{(z^2 - z_1^2)} + i\sqrt{(z_1^2 - 1 + E)}\sqrt{(z_0^2 - z^2)}, \tag{209}$$

so that OC describes a quadric cone, and (α, β) describes an ellipse, as before.

In case III, with $L = 0$, we must put

$$Z = (z_1^2 + z^2)(z_2^2 - z^2), \tag{210}$$

and now

$$\sqrt{(z_1^2 + z_2^2)} \sin\theta e^{\psi i} = \sqrt{(1 - z_2^2)}\sqrt{(z_1^2 + z^2)} + i\sqrt{(1 + z_1^2)}\sqrt{(z_2^2 - z^2)}, \tag{211}$$

$$\sqrt{(z_1^2 + z_2^2)} \frac{F}{An} \rho e^{\varpi i} = \sqrt{(z_2^2 - 1 + E)}\sqrt{(z_1^2 + z^2)} + i\sqrt{(-z_1^2 - 1 + E)}\sqrt{(z_2^2 - z^2)}. \tag{212}$$

The curve of (α, β) is still a conic section, and OC describes a quadric cone relatively to Oz, but the azimuth ψ oscillates between

$$\pm \cos^{-1}\sqrt{\frac{1 - z_2^2}{z_1^2 + z_2^2} z_1^2} = \pm \sin^{-1}\sqrt{\frac{1 + z_1^2}{z_1^2 + z_2^2} z_2^2}, \tag{213}$$

obtained by putting $z = 0$.

Next, with $E = 0$, we have

$$Z = a(z^2 - 1)^2 - 4\left(\frac{Lz - B}{M}\right)^2. \tag{214}$$

With an oblate body, $a = -1$, and Z is always negative, so that no solution exists.

But with a prolate body and $a = +1$,

$$Z = \left(1 - 2\frac{B}{M} + 2\frac{Lz}{M} - z^2\right)\left(1 + 2\frac{B}{M} - 2\frac{Lz}{M} - z^2\right), \tag{215}$$

and

$$\psi - pt = \sin^{-1}\frac{\sqrt{\left(1-2\frac{B}{M}+2\frac{Lz}{M}-z^2\right)}}{\sqrt{2}\sqrt{(1-z^2)}}$$

$$= \cos^{-1}\frac{\sqrt{\left(1+2\frac{B}{M}-2\frac{Lz}{M}-z^2\right)}}{\sqrt{2}\sqrt{(1-z^2)}}$$

$$= \tfrac{1}{2}\sin^{-1}\frac{\sqrt{Z}}{1-z^2}, \tag{216}$$

where

$$\frac{p}{n} = \frac{L}{M}, \text{ and } \frac{dz}{dt} = n\sqrt{Z}, \tag{216a}$$

as may be verified by differentiation.

Also, with $E = 0$,

$$\frac{d\varpi}{dz} = 2\frac{Bz - Lz^2}{M(1-z^2)}\frac{1}{\sqrt{Z}}, \tag{217}$$

so that

$$\frac{d\varpi}{dz} + \frac{d\psi}{dz} = 2\frac{L}{M}\frac{1}{\sqrt{Z}} = 2p\frac{dt}{dz}, \tag{218}$$

$$\varpi + \psi = 2pt + \frac{\pi}{2}; \tag{219}$$

and thus

$$\varpi - pt = \cos^{-1}\frac{\sqrt{\left(1-2\frac{B}{M}+2\frac{Lz}{M}-z^2\right)}}{\sqrt{2}\sqrt{(1-z^2)}}$$

$$= \sin^{-1}\frac{\sqrt{\left(1+2\frac{B}{M}-2\frac{Lz}{M}-z^2\right)}}{\sqrt{2}\sqrt{(1-z^2)}}$$

$$= \tfrac{1}{2}\sin^{-1}\frac{\sqrt{Z}}{1-z^2}. \tag{220}$$

25. With $r = 0$ or $B = 0$, the case of no rotation round OC,

$$w = \omega_a, \tag{221}$$

and

$$Z = a(z^2-1)(z^2-1-aD) - 4\frac{L^2}{M^2}$$

$$= a(z^2-1)(z^2-1-aE) - 4\frac{L^2}{M^2}z^2, \tag{222}$$

and now equations (19) and (12) can be integrated by means of the ordinary Jacobian elliptic functions of the second stage.

When $B=0$, then either N_1 or $N_3=0$, according as the body is prolate or oblate; so that

$$L=\sqrt{(s_1-\sigma)}, \text{ or } -\sqrt{(s_3-\sigma)},$$

and
$$M=\sqrt{(\sigma-s_3)}-\sqrt{(\sigma-s_2)}, \text{ or } \sqrt{(s_1-\sigma)}+\sqrt{(s_2-\sigma)};$$

$$1+aE=\left\{\frac{\sqrt{(\sigma-s_3)}+\sqrt{(\sigma-s_2)}}{\sqrt{(\sigma-s_3)}-\sqrt{(\sigma-s_2)}}\right\}^2 \text{ or } \left\{\frac{\sqrt{(s_1-\sigma)}-\sqrt{(s_2-\sigma)}}{\sqrt{(s_1+\sigma)}+\sqrt{(s_2-\sigma)}}\right\}^2;$$

and

$$Z=\left[z^2-\left\{\frac{\sqrt{(s_1-s_3)}-\sqrt{(s_1-s_2)}}{\sqrt{(\sigma-s_3)}-\sqrt{(\sigma-s_2)}}\right\}^2\right]\left[z^2-\left\{\frac{\sqrt{(s_1-s_3)}+\sqrt{(s_1-s_2)}}{\sqrt{(\sigma-s_3)}-\sqrt{(\sigma-s_2)}}\right\}^2\right]$$

or
$$-\left[z^2-\left\{\frac{\sqrt{(s_1-s_3)}-\sqrt{(s_2-s_3)}}{\sqrt{(s_1-\sigma)}+\sqrt{(s_2-\sigma)}}\right\}^2\right]\left[z^2-\left\{\frac{\sqrt{(s_1-s_3)}+\sqrt{(s_2-s_3)}}{\sqrt{(s_1-\sigma)}+\sqrt{(s_2-\sigma)}}\right\}^2\right].$$

With an oblate body we shall find that this makes $H=0$, $K=1$, $K_1=0,\ldots.$, while in the case of a prolate body the parameter must be of the form (182) for similar reductions to take place; and for the parameter in (185) we shall have $H=K=\frac{1}{2}\sqrt{2}$, when B and $N_1=0$.

26. Passing on to the next simplest pseudo-elliptic case of bisection of a period, by taking a parameter

$$v=\omega_1+\tfrac{1}{2}\omega_3 \tag{223}$$

for a prolate body, and writing χ for $\psi-pt$ throughout, the result must be of the form

$$\chi=\tfrac{1}{2}\cos^{-1}\frac{Hz+H_1}{1-z^2}\sqrt{(z_0-z\,.\,z_3-z)}$$

$$=\tfrac{1}{2}\sin^{-1}\frac{Kz+K_1}{1-z^2}\sqrt{(z-z_2\,.\,z-z_1)} \tag{224}$$

with
$$K^2+H^2=1. \tag{225}$$

The solution must be built up from the associated pseudo-elliptic integral

$$I(\omega_1 + \tfrac{1}{2}\omega_3) = \tfrac{1}{2}\int \frac{(c + c^2 - s) - 2(c + c^2)}{(c + c^2 - s)\sqrt{S}}\, ds$$
$$= \tfrac{1}{2}\sin^{-1}\frac{\sqrt{\{(1+c)^2 - s \,.\, c^2 - s\}}}{c + c^2 - s}$$
$$= \tfrac{1}{2}\cos^{-1}\frac{\sqrt{s}}{c + c^2 - s}\,; \qquad (226)$$

in which

$$s_1 = (1 + c)^2, \quad s_2 = c^2, \qquad s_3 = 0; \qquad (227)$$
$$P(v) = 1, \qquad \sigma = c + c^2, \quad \sqrt{-\Sigma} = 2(c + c^2). \qquad (228)$$

Then, with the preceding notation of (54),

$$N_1^2 = L^2 - 1 - c,$$
$$N_2^2 = L^2 + 0 + c,$$
$$N_3^2 = L^2 + 0 + c + c^2, \qquad (229)$$

so that

$$N_3^2 + N_1^2 N_2^2 = L^4, \qquad (230)$$
$$N_3^2 - L^2 \quad = c + c^2, \qquad (231)$$
$$N_1^2 + N_2^2 \quad = 2L^2 - 1; \qquad (232)$$

and from (59) and (63)

$$\frac{B}{M} = \frac{N_1 N_2 N_3}{LM^2}, \qquad (233)$$

$$M^2 = L^2 - 1 + c + c^2 - \frac{2c + 2c^2}{L},$$
$$LM^2 = 2L^2 - L + (L - 2)N_3^2,$$
$$Mz_0 = - N_1 + N_2 + N_3,$$
$$Mz_3 = + N_1 - N_2 + N_3,$$
$$Mz_2 = + N_1 + N_2 - N_3,$$
$$Mz_1 = - N_1 - N_2 - N_3, \qquad (235)$$

arranged in the order

$$\infty > z_0 > 1 > z_3 > z > z_2 > -1 > z_1 > -\infty;$$

with

$$N_1 < N_2 < N_3.$$

To determine the coefficients H, H_1, K, K_1 in equations (224), differentiate with respect to z; then since

$$(z_0 - z)(z_3 - z) = z^2 - 2\frac{N_3}{M}z + z_0 z_3,$$

$$(z - z_2)(z - z_1) = z^2 + 2\frac{N_3}{M}z + z_1 z_2, \qquad (236)$$

$$\frac{d\chi}{dz} =$$

$$\frac{-\tfrac{1}{2}\dfrac{Hz+H_1}{1-z^2}\sqrt{\left(z^2-2\dfrac{N_3}{M}z+z_0z_3\right)}\left(\dfrac{H}{Hz+H_1}+\dfrac{z-\dfrac{N_3}{M}}{z^2-2\dfrac{N_3}{M}z+z_0z_3}+\dfrac{2z}{1-z^2}\right)}{\left(\dfrac{Kz+K_1}{1-z^2}\right)\sqrt{\left(z^2+2\dfrac{N_3}{M}z+z_1z_2\right)}}$$

$$= \frac{\tfrac{1}{2}\dfrac{Kz+K_1}{1-z^2}\sqrt{\left(z^2+2\dfrac{N_3}{M}z+z_1z_2\right)}\left(\dfrac{K}{Kz+K_1}+\dfrac{z+\dfrac{N_3}{M}}{z^2+2\dfrac{N_3}{M}z+z_1z_2}+\dfrac{2z}{1-z^2}\right)}{\dfrac{Hz+H_1}{1-z^2}\sqrt{\left(z^2-2\dfrac{N_3}{M}z+z_0z_3\right)}}$$

$$= -\tfrac{1}{2}\,\frac{H\left(z^2-2\dfrac{N_3}{M}z+z_0z_3\right)\left(1-z^2\right)+\left(z-\dfrac{N_3}{M}\right)(Hz+H_1)(1-z^2)+2z\,(Hz+H_1)\left(z^2-2\dfrac{N_3}{M}z+z_0z_3\right)}{(Kz+K_1)(1-z^2)\sqrt{Z}}$$

$$= \tfrac{1}{2}\,\frac{K\left(z^2+2\dfrac{N_3}{M}z+z_1z_2\right)(1-z^2)+\left(z+\dfrac{N_3}{M}\right)(Kz+K_1)(1-z^2)+2z\,(Kz+K_1)\left(z^2+2\dfrac{N_3}{M}z+z_1z_2\right)}{(Hz+H_1)(1-z^2)\sqrt{Z}}, \quad (23$$

each of which can be equated to (191), so that

$$(Kz+K_1)\left(\frac{2L-P}{M}z^2-4\frac{B}{M}z+\frac{2L+P}{M}\right),$$

$$= H\left(z^2-2\frac{N_3}{M}z+z_0z_3\right)(z^2-1)+\left(z-\frac{N_3}{M}\right)(Hz+H_1)(z^2-1)-2\,(Hz^2+H_1z)\left(z^2-2\frac{N_3}{M}z+z_0z_3\right)$$

$$= O\,.\,z^4+\left(-H_1+\frac{N_3}{M}H\right)z^3+\left\{-(2+z_0z_3)\,H+3\,\frac{N_3}{M}H_1\right\}z^2+\ldots\ldots, \quad (238$$

$$(Hz+H_1)\left(\frac{2L-P}{M}z^2-4\frac{B}{M}z+\frac{2L+P}{M}\right)$$

$$= -K\left(z^2+2\frac{N_3}{M}z+z_1z_2\right)(z^2-1)-\left(z+\frac{N_3}{M}\right)(Kz+K_1)(z^2-1)+2\,(Kz^2+K_1)\left(z^2+2\frac{N_3}{M}z+z_1z_2\right)$$

$$= 0\,.\,z^4+\left(-K_1-\frac{N_3}{M}K\right)z^3+\left\{(2+z_1z_2)\,K+3\,\frac{N_3}{M}K_1\right\}z^2+\ldots\ldots; \quad (238$$

and equating the coefficients in these identities gives us equations enough, and to spare, to determi H, H_1, K, K_1.

Putting $z = \pm 1$, gives the relations

$$2(K+K_1)\frac{L-B}{M} = -(H+H_1)\left(1 - 2\frac{N_3}{M} + z_0 z_3\right),$$

$$2(H+H_1)\frac{L-B}{M} = \quad (K+K_1)\left(1 + 2\frac{N_3}{M} + z_1 z_2\right),$$

$$2(K-K_1)\frac{L+B}{M} = \quad (H-H_1)\left(1 + 2\frac{N_3}{M} + z_0 z_3\right),$$

$$2(H-H_1)\frac{L+B}{M} = -(K-K_1)\left(1 - 2\frac{N_3}{M} + z_1 z_2\right), \tag{239}$$

which are useful forms to serve for verification.

Working in this manner we find after considerable reduction that

$$\frac{H}{K} = \frac{L^2 - N_1 N_2}{N_3}, \quad \frac{K}{H} = \frac{L^2 + N_1 N_2}{N_3}, \tag{240}$$

and, squaring and adding in (224),

$$K^2 + H^2 = 1, \tag{241}$$

so that

$$\frac{1}{HK} = \frac{K}{H} + \frac{H}{K} = \frac{2L^2}{N_3},$$

or

$$2HK = \frac{N_3}{L^2}, \tag{242}$$

and

$$\frac{K}{H} - \frac{H}{K} = 2\frac{N_1 N_2}{N_3},$$

so that

$$K^2 - H^2 = \frac{N_1 N_2}{L^2}; \tag{243}$$

$$K^2 = \frac{L^2 + N_1 N_2}{2L^2},$$

$$H^2 = \frac{L^2 - N_1 N_2}{2L^2}. \tag{244}$$

Also

$$\frac{H_1}{K} = \quad \frac{(L+1)^2 - N_1 N_2}{M},$$

$$\frac{K_1}{H} = \frac{-(L-1)^2 - N_1 N_2}{M}, \tag{245}$$

so that

$$\frac{H_1}{K} + \frac{K_1}{H} = -2\frac{N_1 N_2}{M}, \tag{246}$$

as is verified by squaring and adding; and all other relations are also found to verify.

27. In the case of the oblate body with parameter

$$v = \tfrac{1}{3}\omega_3, \tag{247}$$

we must take

$$\begin{aligned}\chi &= \tfrac{1}{2}\cos^{-1}\frac{Hz + H_1}{1 - z^3}\sqrt{(z - z_1 . z - z_2)} \\ &= \tfrac{1}{2}\sin^{-1}\frac{Kz + K_1}{1 - z^3}\sqrt{(z_3 - z . z - z_0)},\end{aligned} \tag{248}$$

and build up upon the pseudo-elliptic integral

$$\begin{aligned}I(\tfrac{1}{3}\omega_3) &= \tfrac{1}{2}\int\frac{(1 + 2c)(s + c + c^2) - 2(1 + 2c)(c + c^2)}{(s + c + c^2)\sqrt{S}}\,ds \\ &= \tfrac{1}{2}\sin^{-1}\frac{\sqrt{\{s - (1 + c)^2 . s - c^2\}}}{s + c + c^2} \\ &= \tfrac{1}{2}\cos^{-1}\frac{(1 + 2c)\sqrt{s}}{s + c + c^2},\end{aligned} \tag{249}$$

so that, with the same s_1, s_2, s_3, we take

$$P(v) \text{ or } P = 1 + 2c, \quad \sigma = -c - c^2, \quad \sqrt{-\Sigma} = 2(1 + 2c)(c + c^2); \tag{250}$$

and now, with (52),

$$\begin{aligned}N_\alpha^2 &= s_\alpha - \sigma - L^2, \\ N_1^2 &= 1 + 3c + 2c^2 - L^2, \\ N_2^2 &= c + 2c^2 - L^2, \\ N_3^2 &= c + c^2 - L^2,\end{aligned} \tag{251}$$

so that

$$N_1^2 + N_2^2 = P^2 - 2L^2, \tag{252}$$

$$N_3^2 + L^2 = c + c^2, \tag{253}$$

$$N_1^2 N_2^2 - L^4 = P^2 N_3^2; \tag{254}$$

and from (59) and (63),

$$\frac{B}{M} = -\frac{N_1 N_2 N_3}{LM^2}, \tag{255}$$

$$\begin{aligned}M^2 &= 1 + 5c + 5c^2 - L^2 - \frac{2(1 + 2c)(c + c^2)}{L} \\ &= P^2 + N_3^2 - 2\frac{P}{L}(N_3^2 + L^2).\end{aligned} \tag{256}$$

Proceeding as before, we now find

$$\frac{H}{K} = \frac{N_1 N_2 + L^2}{PN_3}, \quad \frac{K}{H} = \frac{N_1 N_2 - L^2}{PN_3}, \tag{257}$$

and

$$H^2 - K^2 = 1, \tag{258}$$

$$2HK = \frac{PN_3}{L^2}, \tag{259}$$

$$H^2 + K^2 = \frac{N_1 N_2}{L^2}, \tag{260}$$

$$H^2 = \frac{N_1 N_2 + L^2}{2L^2},$$

$$K^2 = \frac{N_1 N_2 - L^2}{2L^2}, \tag{261}$$

$$\frac{H_1}{K} = \frac{(L-P)^2 + N_1 N_2}{PM},$$

$$\frac{K_1}{H} = \frac{(L-P)^2 - N_1 N_2}{PM}, \tag{262}$$

and now all the conditions are found to be satisfied.

28. Both results for the prolate and oblate body with parameter

$$v = \omega_1 + \tfrac{1}{2}\omega_3, \text{ or } \tfrac{1}{2}\omega_3, \tag{263}$$

can be incorporated in the form

$$\chi = \tfrac{1}{2}\cos^{-1}\frac{Hz + H_1}{1 - z^2}\sqrt{(z - z_0 \,.\, z - z_3)} \text{ or } \sqrt{(z - z_1 \,.\, z - z_2)}$$

$$= \tfrac{1}{2}\sin^{-1}\frac{Kz + K_1}{1 - z^2}\sqrt{(z - z_1 \,.\, z - z_2)} \text{ or } \sqrt{(z_3 - z \,.\, z - z_0)}, \tag{264}$$

with

$$H^2 + aK^2 = 1, \tag{265}$$

$$H^2 - aK^2 = -a\frac{N_1 N_2}{L^2}, \tag{266}$$

$$2HK = \frac{PN_3}{L^2}. \tag{267}$$

If we try to cancel the secular term pt by putting

$$L = \tfrac{1}{2}P = \tfrac{1}{2}, \text{ or } \tfrac{1}{2}(1 + 2c), \tag{268}$$

we find

$$N_1^2 = -\tfrac{3}{4} - c, \text{ or } N_3^2 = -\tfrac{1}{4}, \tag{269}$$

thus leading to imaginary results.

As verifications we may take the numerical case for a parameter $v=\omega_1+\frac{1}{2}\omega_3$, worked out on p. 534 of "Les fonctions elliptiques et leur applications," Greenhill-Griess, 1895.

For a parameter $v=\frac{1}{2}\omega_3$, take, as a special case,

$$B=0,\quad N_3=0,\quad N_1=1+c,\quad N_2=c,$$
$$L=-\sqrt{(c+c^2)},\quad M=\sqrt{(1+2c)}\{\sqrt{(1+c)}+\sqrt{c}\},$$
$$\frac{p}{n}=\frac{L-\frac{1}{2}P}{M}=-\frac{\sqrt{(1+c)}+\sqrt{c}}{2\sqrt{(1+2c)}};$$

and now

$$\psi-pt=\tfrac{1}{2}\cos^{-1}\frac{z}{1-z^2}\sqrt{\left[z^2-\frac{\{\sqrt{(1+c)}-\sqrt{c}\}^2}{1+2c}\right]}$$
$$=\tfrac{1}{2}\sin^{-1}\frac{\dfrac{\sqrt{(1+c)}+\sqrt{c}}{\sqrt{(1+2c)}}}{1-z^2}\sqrt{[(1+2c)\{\sqrt{(1+c)}-\sqrt{c}\}^2-z^2]}. \tag{269a}$$

29. Proceeding to the next case where the parameter

$$v=\omega_1+\tfrac{1}{3}\omega_3,\ \text{or}\ \tfrac{1}{3}\omega_3, \tag{270}$$

both results for the prolate and the oblate body can be incorporated in the form

$$\chi=\tfrac{1}{3}\cos^{-1}\frac{Hz^2+H_1z+H_2}{(1-z^2)^{\frac{3}{2}}}\sqrt{(z_3-z\,.\,z-z_1)}\ \text{or}\ \sqrt{(z-z_2\,.\,z-z_1)}$$
$$=\tfrac{1}{3}\sin^{-1}\frac{Kz^2+K_1z+K_2}{(1-z^2)^{\frac{3}{2}}}\sqrt{(z_0-z\,.\,z-z_2)}\ \text{or}\ \sqrt{z_3-z\,.\,z-z_0}). \tag{271}$$

With a parameter

$$v=\omega_1+\tfrac{2}{3}\omega_3 \tag{272}$$

we should have to interchange the suffixes 3 and 0, in the first form.

In the associated pseudo-elliptic integrals (Proc. London Math. Society, XXV, p. 218), corresponding to the parameters in (270),

$$s_1=(1-c)^3,\quad s_2=c^3,\quad s_3=(c-c^2)^2; \tag{273}$$
$$P=\tfrac{2}{3}(2-c)(1-2c),\ \text{or}\ \tfrac{2}{3}(1+c)(2-c), \tag{274}$$
$$\sigma=2c(1-c)^2,\ \text{or}\ -2c+2c^3, \tag{275}$$
$$\sqrt{-\Sigma}=2c(1-c)^2(2-c)(1-2c),\ \text{or}\ 2c(1-c)(1+c)(2-c). \tag{276}$$

Differentiating as before, and equating coefficients, will serve to determine the unknown coefficients H and K; the work, which is very long and laborious, has been carried out for me by Mr. T. I. Dewar, and he has found that

$$K^2 - aH^2 = \tfrac{3}{2}\frac{PN_1N_3}{L^3} \text{ or } \tfrac{3}{2}\frac{PN_1N_2}{L^2}, \tag{277}$$

while squaring and adding in (271) gives

$$K^2 + aH^2 = 1, \tag{278}$$

whence K and H being determined, the other coefficients readily follow.

Thus, for the prolate body, $v = \omega_1 + \frac{1}{3}\omega_3$,

$$H^2 = \frac{L^3 - (2-c)(1-2c)\,N_1N_3}{2L^3},$$

$$K^2 = \frac{L^3 + (2-c)(1-2c)\,N_1N_3}{2L^3}; \tag{279}$$

so that

$$2HK = \frac{L^2 - (1-c)^2(2-c)(1-2c)}{L^3}\,N_2; \tag{280}$$

while for the oblate body, $v = \frac{1}{3}\omega_3$,

$$H^2 = \frac{(1+c)(2-c)\,N_1N_2 - L^3}{2L^3},$$

$$K^2 = \frac{(1+c)(2-c)\,N_1N_2 - L^3}{2L^3}; \tag{281}$$

so that

$$2HK = \frac{(1+c)(2-c) - L^2}{L^3}\,N_3. \tag{282}$$

If we try to cancel the secular term by putting

$$L = \tfrac{1}{2}P = \tfrac{1}{3}(2-c)(1-2c), \text{ or } \tfrac{1}{3}(1+c)(2-c), \tag{283}$$

we find

$$9N_1^2 = (1-2c)(1-c)(-5+2c+2c^2), \text{ or } 9N_3^2 = -2(1+c)(2-c)(1-2c)^2, \tag{284}$$

and these are negative for the region $0 < c < 1$, so that algebraical cases cannot be constructed.

As a numerical verification with a parameter $v = \omega_1 + \frac{1}{3}\omega_3$, we may take the case worked out in the "Applications of Elliptic Functions," p. 348, in which

$$\psi - nt = \tfrac{1}{3}\cos^{-1}\frac{\sqrt{7}z^2 - 4z + \sqrt{7}}{2\sqrt{2}\,(1-z^2)^{\frac{3}{2}}}\sqrt{(-z^2 - 2\sqrt{7}z + 5)}$$

$$= \tfrac{1}{3}\sin^{-1}\frac{(-z^2 + 2\sqrt{7}z - 3)^{\frac{3}{2}}}{2\sqrt{2}\,(1-z^2)^{\frac{3}{2}}}. \tag{284a}$$

With $B=0$ and a parameter $v=\omega_1+\frac{2}{3}\omega_3$,

$$
\begin{aligned}
&N_1=0, \quad N_2=\sqrt{(1-2c)}, \quad N_3=(1-c)\sqrt{(1-c^2)},\\
&L=\sqrt{(1-2c\,.\,1-c^2)}, \quad M=c\sqrt{(1-c^2)}-c\sqrt{(1-2c)},\\
&P=\tfrac{2}{3}(1+c)(1-2c),
\end{aligned}
$$

$$
\begin{aligned}
\psi-pt&=\tfrac{1}{3}\cos^{-1}\frac{Hz^2+H_1z+H_2}{(1-z^3)^{\frac{3}{2}}}\sqrt{\left\{\left(\frac{N_2-N_3}{M}\right)^2-z^3\right\}}\\
&=\tfrac{1}{3}\sin^{-1}\frac{Kz^2+K_1z+K_2}{(1-z^3)^{\frac{3}{2}}}\sqrt{\left\{\left(\frac{N_2+N_3}{M}\right)^2-z^3\right\}},
\end{aligned}
\tag{284b}
$$

in which $H=0, \quad K=1, \quad K_1=0, \quad H_2=0,$

$$
H_1=3\,\frac{p}{n}=\frac{(1+c)\sqrt{(1-2c)}+(1-2c)\sqrt{(1-c)}}{c^2},
$$

$$
K_2=\frac{-M}{N_2-N_3}=-\tfrac{1}{2}\left\{\frac{\sqrt{(1-c)}-\sqrt{(1+c\,.\,1-2c)}}{c}\right\}^2.
$$

With $B=0$, and a parameter $v=\frac{1}{3}\omega_3$,

$$
\begin{aligned}
&N_3=0, \quad L=-\sqrt{(1-c^2\,.\,2c-c^2)},\\
&N_1^2=(1-c)^3(1+c), \quad N_2^2=c^2(2c-c^2),\\
&M=\sqrt{(1-c^2)}+\sqrt{(2c-c^2)},
\end{aligned}
$$

$$
\begin{aligned}
\psi-pt&=\tfrac{1}{3}\cos^{-1}\frac{Hz^2+H_1z+H_2}{(1-z^3)^{\frac{3}{2}}}\sqrt{\left\{z^3-\left(\frac{N_1-N_2}{M}\right)^2\right\}}\\
&=\tfrac{1}{3}\sin^{-1}\frac{Kz^2+K_1z+K_2}{(1-z^3)^{\frac{3}{2}}}\sqrt{\left\{\left(\frac{N_1+N_2}{M}\right)^2-z^3\right\}},
\end{aligned}
\tag{284c}
$$

in which

$$
H=0, \quad H_1=-3\,\frac{p}{n}=(2-c)\sqrt{(1-c^2)}+(1+c)\sqrt{(2c-c^2)}, \quad H_2=0;
$$

$$
K=1, \quad K_1=0, \quad K_2=\frac{M}{N_1+N_2}=\tfrac{1}{2}\{\sqrt{(1-c^2)}+\sqrt{(2c-c^2)}\}^2.
$$

30. These algebraical calculations are too long and complicated to be inserted here, and the complexity compelled us, as in the corresponding work for the Spherical Catenary, in the Proc. London Math. Society, XXVII, to turn elsewhere for some clue to the values of the leading coefficients H and K, upon which the remainder depend by simple linear relations.

As in the Spherical Catenary, we turn to the degenerate form assumed when we take

$$
z=\infty, \quad u=v_3.
$$

The preceding cases enable us to infer that the general form of the solution for a parameter

$$v = \omega_1 + \frac{2\omega_3}{n}, \text{ or } \omega_1 + \frac{\omega_3}{n} \text{ or } \frac{\omega_3}{n}, \tag{285}$$

where n is odd, can be written

$$\begin{aligned}
\chi &= \frac{1}{n}\cos^{-1}\frac{Hz^{n-1}+\ldots\ldots}{(1-z^3)^{\frac{1}{2}n}}\sqrt{(z_0-z\,.\,z-z_1)}, \text{ or } \sqrt{(z_3-z\,.\,z-z_1)}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{or } \sqrt{(z-z_1\,.\,z-z_2)} \\
&= \frac{1}{n}\sin^{-1}\frac{Kz^{n-1}+\ldots\ldots}{(1-z^3)^{\frac{1}{2}n}}\sqrt{(z_3-z\,.\,z-z_2)}, \text{ or } \sqrt{(z_0-z\,.\,z-z_2)}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{or } \sqrt{(z_3-z\,.\,z-z_0)},
\end{aligned} \tag{286}$$

but for parameters

$$v = \omega_1 + \frac{\omega_3}{n}, \text{ or } \frac{\omega_3}{n}, \tag{287}$$

and n even, we must take

$$\begin{aligned}
\chi &= \frac{1}{n}\cos^{-1}\frac{Hz^{n-1}+\ldots\ldots}{(1-z^3)^{\frac{1}{2}n}}\sqrt{(z_0-z\,.\,z_3-z)}, \text{ or } \sqrt{(z-z_1\,.\,z-z_2)} \\
&= \frac{1}{n}\sin^{-1}\frac{Kz^{n-1}+\ldots\ldots}{(1-z^3)^{\frac{1}{2}n}}\sqrt{(z-z_2\,.\,z-z_1)}, \text{ or } \sqrt{(z_3-z\,.\,z-z_0)}.
\end{aligned} \tag{288}$$

Squaring and adding in (286) and (288) shows that

$$H^2 + K^2 = 1, \text{ or } H^2 - K^2 = (-1)^n, \tag{289}$$

according as the body is prolate or oblate; and putting $z = \infty$,

$$\begin{aligned}
\chi &= \frac{1}{n}\tan^{-1}\frac{K\sqrt{a}}{H} = \frac{1}{2n}\sin^{-1}2HK\sqrt{a} \\
&= \frac{1}{2n}\cos^{-1}(H^2-K^2), \text{ or } \frac{1}{2n}\cos^{-1}(H^2+K^2)(-1)^n.
\end{aligned} \tag{290}$$

But, by means of the formula

$$\frac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)} = \frac{\sigma(2u-v_1-v_2)\,\sigma^2(u+v_1)\,\sigma^2(u+v_2)}{\sigma(2u+v_1+v_2)\,\sigma^2(u-v_1)\,\sigma^2(u-v_2)}, \tag{291}$$

we may write equation (133)

$$\begin{aligned}
\psi i &= \tfrac{1}{2}Qnt + \tfrac{1}{4}\log\frac{\sigma(2u+v)}{\sigma(2u-v)} + \tfrac{1}{4}\log\frac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)} \\
&= ipt + \tfrac{1}{2}iI(2u, v) \qquad\quad + \tfrac{1}{4}\log\frac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)},
\end{aligned} \tag{292}$$

thus adding the amplitudes and parameters of the two Elliptic Integrals of the Third Kind in ψ_1 and ψ_2.

Putting $z = \infty$ and $u = v_3$ in this last relation, then from (96), (97),

$$\frac{\wp(v_3 - v_1) - \wp(v_3 - v_2)}{\wp(v_3 + v_1) - \wp(v_3 + v_2)} = -1, \tag{293}$$

while $I(2v_3, v)$ is the value of the pseudo-elliptic integral when $u = 2v_3$ or

$$\frac{s - \sigma}{M^2} = \wp 2v_3 - \wp(v_1 + v_2) = \frac{L^2}{M^2}, \tag{294}$$

$$s - \sigma = L^2, \tag{295}$$

$$s - s_\alpha = aN_\alpha^2. \tag{296}$$

The form of $I(v)$ being given by

$$\begin{aligned} I(v) &= \frac{1}{n}\sin^{-1}\frac{F\sqrt{(s - s_\alpha)}}{(s - \sigma)^{\frac{1}{2}n}} \\ &= \frac{1}{n}\cos^{-1}\frac{G\sqrt{(s - s_\beta \,.\, s - s_\gamma)}}{(s - \sigma)^{\frac{1}{2}n}}, \end{aligned} \tag{297}$$

therefore, when $z = \infty$ and $\psi - pt$ is replaced by χ,

$$\chi = \tfrac{1}{2} I(2v_3, v) + \tfrac{1}{4} i \log(-1); \tag{298}$$

or, disregarding for a moment the ambiguities of sign,

$$\frac{1}{2n}\sin^{-1}\frac{FN_\alpha\sqrt{a}}{L^n} = \frac{1}{2n}\sin^{-1} 2HK\sqrt{a}, \tag{299}$$

$$\frac{1}{2n}\cos^{-1}\frac{GN_\beta N_\gamma}{L^n} = \frac{1}{2n}\cos^{-1}(H^2 - aK^2), \tag{300}$$

or

$$2HK = \frac{FN_\alpha}{L^n} \tag{301}$$

$$H^2 - aK^2 = \pm\frac{GN_\beta N_\gamma}{L^n}, \tag{302}$$

and these equations, together with (289), determine the leading coefficients H and K, upon which the other coefficients depend by simple linear relations.

Since $z = \infty$ in (167) makes $\varpi = \psi$, we see that the leading coefficients I and J in (184) are determined at the same time, by taking $I = K$ and $J = H$ or $I = H$ and $J = K$, according to circumstances.

31. To make quite sure of the plus and minus signs, which are apt to be baffling, it is well to make a recapitulation of all the various cases which may occur, according as the body is prolate or oblate, and n is even or odd.

When the body is prolate, and the parameter of the form

$$v = \omega_1 + \frac{\omega_3}{n}, \tag{303}$$

then, when n is even, we must take

$$\chi = \frac{1}{n}\cos^{-1}\frac{Hz^{n-1} + H_1 z^{n-2} + \ldots + H_{n-1}}{(1-z^2)^{\frac{1}{2}n}}\sqrt{(z_0 - z \,.\, z_3 - z)}$$
$$= \frac{1}{n}\sin^{-1}\frac{Kz^{n-1} + K_1 z^{n-2} + \ldots + K_{n-1}}{(1-z^2)^{\frac{1}{2}n}}\sqrt{(z - z_2 \,.\, z - z_1)}, \tag{304}$$

and when n is odd,

$$\chi = \frac{1}{n}\cos^{-1}\frac{Hz^{n-1} \qquad + \ldots + H_{n-1}}{(1-z^2)^{\frac{1}{2}n}}\sqrt{(z_3 - z \,.\, z - z_1)}$$
$$= \frac{1}{n}\sin^{-1}\frac{Kz^{n-1} \qquad + \ldots + K_{n-1}}{(1-z^2)^{\frac{1}{2}n}}\sqrt{(z_0 - z \,.\, z - z_2)}. \tag{305}$$

But with a parameter

$$v = \omega_1 + \frac{2\omega_3}{n}, \tag{306}$$

where n is an odd number,

$$\chi = \frac{1}{n}\cos^{-1}\frac{Hz^{n-1} + \ldots}{(1-z^2)^{\frac{1}{2}n}} \qquad \sqrt{(z_0 - z \,.\, z - z_1)}$$
$$= \frac{1}{n}\sin^{-1}\frac{Kz^{n-1} + \ldots}{(1-z^2)^{\frac{1}{2}n}} \qquad \sqrt{(z_3 - z \,.\, z - z_2)}, \tag{307}$$

obtainable from the preceding case (306) by an interchange of z_0 and z_3, or z_1 and z_2.

In all these cases of the prolate body

$$z_3 > z > z_2, \tag{308}$$

and
$$H^2 + K^2 = 1. \tag{309}$$

Putting $z = \infty$,

$$\chi = \frac{1}{n}\tan^{-1}\frac{K}{H} = \frac{1}{2n}\sin^{-1}2HK = \frac{1}{2n}\cos^{-1}(H^2 - K^2), \tag{310}$$

and at the same time

$$\chi = \tfrac{1}{2}I(2v_3, v) = \frac{1}{2n}\sin^{-1}\frac{Fn_\alpha}{L^n} = \frac{1}{2n}\cos^{-1}\frac{GN_\beta N_\gamma}{L^n}; \tag{311}$$

so that

$$2HK = \frac{FN_\alpha}{L^n}, \tag{312}$$

$$H^2 - K^2 = \frac{GN_\beta N_\gamma}{L^n}. \tag{313}$$

32. When the body is oblate and the parameter is of the form

$$v = \frac{\omega_3}{n}, \tag{314}$$

we must take

$$\begin{aligned}\chi &= \frac{1}{n}\cos^{-1}\frac{Hz^{n-1} + \dots + H_{n-1}}{(1-z^2)^{\frac{1}{2}n}}\sqrt{(z_3 - z \,.\, z - z_0)} \\ &= \frac{1}{n}\sin^{-1}\frac{Kz^{n-1} + \dots + K_{n-1}}{(1-z^2)^{\frac{1}{2}n}}\sqrt{(z - z_2 \,.\, z - z_1)}\end{aligned} \tag{315}$$

and

$$H^2 - K^2 = (-1)^n. \tag{316}$$

Putting $z = \infty$ and $u = v_3$,

$$\chi = \frac{1}{n}\tan^{-1}\frac{Ki}{H} = \frac{1}{2n}\sin^{-1}2HKi(-1)^n = \frac{1}{2n}\cos^{-1}(H^2+K^2)(-1)^n, \tag{317}$$

so that, allowing for the effect of the $\frac{1}{2}i\log(-1)$ in (298), we may put

$$2HK = \frac{FN_\alpha}{L^n}, \tag{318}$$

$$H^2 + K^2 = \frac{GN_\alpha}{L^n}. \tag{319}$$

33. Finally in the case of an oblate body, when the parameter

$$v = \frac{2\omega_3}{n}, \tag{320}$$

and n is an odd number, we must take

$$\begin{aligned}\chi &= \frac{1}{n}\cos^{-1}\frac{Hz^n + H_1z^{n-1} + \dots + H_n}{(1-z^2)^{\frac{1}{2}n}} \\ &= \frac{1}{n}\sin^{-1}\frac{Kz^{n-2} + K_1z^{n-3} + \dots + K_{n-2}}{(1-z^2)^{\frac{1}{2}n}}\sqrt{Z},\end{aligned} \tag{321}$$

with

$$K^2 - H^2 = 1; \tag{322}$$

and proceeding to $z = \infty$,

$$\chi = \frac{1}{n}\tan^{-1}\frac{Ki}{H} = \frac{1}{2n}\cos^{-1}(H^2 + K^2). \tag{323}$$

At the same time the associated pseudo-elliptic integral

$$I(2v_3,\ v) = \frac{1}{n}\cos^{-1}\frac{F}{L^n} = \frac{1}{n}\sin^{-1}\frac{GN_\alpha N_\beta N_\gamma i}{L^n}, \tag{324}$$

so that

$$H^2 + K^2 = \frac{F}{L^n}, \tag{325}$$

$$2HK = \frac{GN_\alpha N_\beta N_\gamma}{L^n}. \tag{326}$$

In such cases it is generally simpler to take a parameter

$$v = \frac{4\omega_3}{n} \tag{327}$$

so as to make $\sigma = 0$ in (294), and now, when $z = \infty$,

$$s = L^2, \quad N_\alpha^2 = s_\alpha - L^2. \tag{328}$$

The resolution of the cubic S in (50) is not required now, because symmetric functions only of the roots s_1, s_2, s_3 occur; thus from (63),

$$\begin{aligned} M^2 &= N_1^2 + N_2^2 + N_3^2 + 2L^2 - \frac{\sqrt{-\Sigma}}{L} \\ &= s_1 + s_2 + s_3 - L^2 - \frac{xy}{L} \\ &= \tfrac{1}{4}(y+1)^2 - 2x - L^2 - \frac{xy}{L}, \end{aligned} \tag{329}$$

$$\begin{aligned} LM^2 &= -xy + \tfrac{1}{4}\{(y+1)^2 - 8x\}L - L^3 \\ &= M^3 i\wp' v + 3LM^2\wp v - L^3, \end{aligned} \tag{330}$$

as in Lamé's equation of the second order.

34. With $s = L^2$,

$$-S = \{xy + (y+1)L^2\}^2 - 4L^2(x+L^2)^2 = S_1 S_2, \tag{331}$$

where

$$\begin{aligned} S_1 &= xy + 2xL + (y+1)L^2 + 2L^3, \\ S_2 &= xy - 2xL + (y+1)L^2 - 2L^3. \end{aligned} \tag{332}$$

At the same time we shall find that we can put

$$\begin{aligned} \tfrac{1}{2}I(2v_3,\ v) &= \frac{1}{n}\sin^{-1}\frac{A + BL + CL^2 \dots}{2L^{\frac{1}{2}n}}\sqrt{S_1} \\ &= \frac{1}{n}\cos^{-1}\frac{A - BL + CL^2 \dots}{2L^{\frac{1}{2}n}}\frac{\sqrt{S_2}}{i}; \end{aligned} \tag{333}$$

while with $z = \infty$,

$$\chi = \frac{1}{n} \sin^{-1} K = \frac{1}{n} \cos^{-1} \frac{H}{i}, \tag{334}$$

so that

$$K \text{ or } H = \frac{A + BL + CL^2 + \dots}{2L^{\frac{1}{2}n}, \text{ or } 2(-L)^{\frac{1}{2}n}} \sqrt{S_1},$$

$$H \text{ or } K = \frac{A - BL + CL^2 - \dots}{2L^{\frac{1}{2}n} \text{ or } 2(-L)^{\frac{1}{2}n}} \sqrt{S_2}, \tag{335}$$

thus determining the leading coefficients, when the parameter $v = 4\omega_3/n$.

35. For instance, with $n = 3$, and taking the corresponding pseudo-elliptic integral, with $P = \frac{1}{3}$,

$$I(\tfrac{4}{3}\omega_3) = \tfrac{1}{2}\int^s \frac{\frac{1}{3}s + c}{s\sqrt{\{4s^3 - (s+c)^2\}}}$$

$$= \tfrac{1}{3} \cos^{-1} \frac{s - c}{2s^{\frac{3}{2}}} = \tfrac{1}{3} \sin^{-1} \frac{\sqrt{S}}{2s^{\frac{3}{2}}}, \tag{336}$$

we have

$$K = \frac{\sqrt{(c + L^2 + 2L^3)}}{2L^{\frac{3}{2}}},$$

$$H = \frac{\sqrt{(c + L^2 - 2L^3)}}{2L^{\frac{3}{2}}}. \tag{337}$$

Then

$$LM^2 = c + \tfrac{1}{4}L - L^3, \tag{338}$$

$$\frac{B}{M} = -\frac{N_1 N_2 N_3}{LM^2} = \frac{\frac{1}{2}\sqrt{\{(c+L^2)^2 - 4L^6\}}}{c + \frac{1}{4}L - L^3}$$

$$= 2\frac{\sqrt{\{(c + L^2)^2 - 4L^6\}}}{4c + L - 4L^3}, \tag{339}$$

and the result is of the form

$$\psi - pt = \tfrac{1}{3} \cos^{-1} \frac{Hz^3 + H_1 z^2 + H_2 z + H_3}{(1 - z^2)^{\frac{3}{2}}} = \tfrac{1}{3} \sin^{-1} \frac{Kz + K_1}{(1 - z^2)^{\frac{3}{2}}} \sqrt{Z}. \tag{340}$$

By squaring and adding, and by differentiation, we find

$$H_1 = \frac{1 - 6L}{2M} K,$$

$$K_1 = -\frac{H}{K} H_1 = -\frac{1 - 6L}{M} H,$$

$$H_2 = -\frac{1 + 6L}{2M} K_1 = \frac{1 - 36L^2}{2M^2} H,$$

$$H_3 = \frac{2L - P}{M} K - \frac{2L + P}{2M} K + 2\frac{B}{M} K_1,$$

$$= \frac{2L - 3P}{2M} K + 2\frac{B}{M} K_1$$

$$= -\frac{1 - 2L}{2M} K - 4\frac{\sqrt{\{(c + L^2)^2 - 4L^6\}}}{4c + L - 4L^3} \frac{1 - 6L}{M} H$$

$$= -\left\{\frac{1 - 2L}{M} + 4\frac{c + L^2 - 2L^3}{4c + L - 4L^3} \frac{1 - 6L}{M}\right\} K$$

$$= -\left\{1 - 2L + 4(1 - 6L)\frac{c + L^2 - 2L^3}{4c + L - 4L^3}\right\} \frac{K}{M}. \tag{341}$$

If we cancel the secular term by putting

$$L = \tfrac{1}{2} P = \tfrac{1}{6}, \tag{342}$$

then $$H_1 = 0, \quad K_1 = 0, \quad H_2 = 0,$$

and
$$H_3 = -\tfrac{1}{3}\frac{K}{M},$$
$$H = \sqrt{(54c+1)},$$
$$K = \sqrt{(54c+2)},$$
$$9M^2 = 54c + 2 = K^2,$$
$$3M = -K, \quad H_3 = -1,$$
$$\frac{B}{M} = \tfrac{1}{2}\frac{H}{K}, \quad \frac{L}{M} = \frac{1}{2K}, \tag{343}$$

and now

$$\psi = \tfrac{1}{3}\cos^{-1}\frac{\sqrt{(54c+1)}\,z^3 - 1}{(1-z^2)^{\frac{3}{2}}}$$
$$= \tfrac{1}{3}\sin^{-1}\frac{z}{(1-z^2)^{\frac{3}{2}}}\sqrt{\{-(54c+2)z^4 + 3z^2 + 2\sqrt{(54c+1)}\,z - 3\}}. \tag{344}$$

This can also be written, with H for $\sqrt{(54c+1)}$,

$$(1-z^2)^{\frac{3}{2}} e^{3\psi i} = Hz^3 - 1 + iz\sqrt{\{-(H^2+1)z^4 + 3z^2 + 2Hz - 3\}}, \tag{345}$$

which by differentiation will be found to verify the relation

$$\frac{d\psi}{dz} = 2\frac{L - Bz}{M(1-z^2)\sqrt{Z_1}}, \tag{346}$$

and thus (345) represents an algebraical case of the motion of the axis OC of an oblate body, relatively to a line Oz fixed in direction.

At the same time the curve of (α, β) or (ρ, ϖ) will be given by a relation of the form

$$\varpi = \tfrac{1}{3}\cos^{-1}\frac{Iz^3 + I_1 z^2 + I_2 z + I_3}{(z^2 - 1 + E)^{\frac{3}{2}}}$$
$$= \tfrac{1}{3}\sin^{-1}\frac{Jz + J_1}{(z^2 - 1 + E)^{\frac{3}{2}}}\sqrt{Z}, \tag{347}$$

where $$I = K = \sqrt{(H^2+1)} \text{ and } J = H, \tag{348}$$

or
$$\left(\frac{F}{An}\rho e^{\varpi i}\right)^3 = \left\{\frac{F}{An}(\alpha + \beta i)\})\right\}^3$$
$$= Iz^3 + I_1 z^2 + I_2 z + I_3 + i(Jz + J_1)\sqrt{Z}, \tag{349}$$

and we shall find that with

$$E = 2\frac{H^2-1}{H^2+1},\quad I_1 = \frac{3H}{\sqrt{(H^2+1)}},\quad I_2 = 0,\quad I_3 = -\frac{H^3+9H}{(H^2+1)^{\frac{3}{2}}},\quad J_1 = 3, \tag{350}$$

the differential relation of (169) can be satisfied.

The discriminant of

$$Z_1 = -(H^2+1)z^4 + 3z^2 + 2Hz - 3 \tag{351}$$

is

$$-\tfrac{27}{16}(H^2-1)^3(H^2+1). \tag{352}$$

If $H^2 < 1$, all four roots of this quartic are imaginary, and Z is always negative, so that no real solution exists.

But if $H^2 > 1$, the discriminant is negative, and Z has two real roots, so that real cases of motion can be constructed.

36. Making use of the results of the pseudo-elliptic integrals of higher orders, we find for $n = 5$, putting Halphen's $y = x$,

$$K \text{ or } H = \frac{(x-L)\sqrt{\{x^2 + 2xL + (1+x)L^2 + 2L^3\}}}{2L^{\frac{3}{2}}, \text{ or } 2(-L)^{\frac{3}{2}}},$$

$$H \text{ or } K = \frac{(x+L)\sqrt{\{x^2 - 2xL + (1+x)L^2 - 2L^3\}}}{2L^{\frac{3}{2}} \text{ or } 2(-L)^{\frac{3}{2}}}, \tag{353}$$

according as L is positive or negative, and to make the solution algebraical by cancelling the secular term, put

$$L = \tfrac{1}{2}P(v) = \frac{1-3x}{10}. \tag{354}$$

This makes

$$K \text{ or } H = \frac{(-1+13x)\sqrt{(3+33x+121x^2+9x^3)}}{(1-3x)^{\frac{3}{2}} \text{ or } (3x-1)^{\frac{3}{2}}},$$

$$H \text{ or } K = \frac{(1+7x)\sqrt{\{2(1-18x)(1-11x-x^2)}}{(1-3x)^{\frac{3}{2}} \text{ or } (3x-1)^{\frac{3}{2}}}; \tag{355}$$

$$N_1N_2N_3 = \sqrt{\{(s_1-L^2)(s_2-L^2)(s_3-L^2)\}}$$
$$= \frac{\sqrt{\{2(1-18x)(1-11x-x^2)(3+33x+121x^2+9x^3)\}}}{1000}, \tag{356}$$

$$LM^2 = \frac{24(1+2x)(1-11x-x^2)}{1000}, \tag{357}$$

and putting

$$\psi = \tfrac{1}{5}\cos^{-1}\frac{Hz^5 + H_1z^4 + \dots + H_5}{(1-z^2)^{\frac{5}{2}}}$$
$$= \tfrac{1}{5}\sin^{-1}\frac{Kz^3 + K_1z^2 + K_2 + K_3}{(1-z^2)^{\frac{5}{2}}}\sqrt{Z}, \tag{358}$$

where $$Z = -(z^2-1)(z^2-1+E) - 4\left(\frac{Lz-B}{M}\right)^2, \tag{359}$$

the differential relation (46) shows at once that
$$H_1 = 0 \text{ and } K_1 = 0. \tag{360}$$

As the result of an algebraical verification, it will be found that with

$$\begin{aligned}
L &= -\frac{3x-1}{10},\\
LM^2 &= -\frac{24(2x+1)(x^2+11x-1)}{1000},\\
M^2 &= \frac{24(2x+1)(x^2+11x-1)}{100(3x-1)},\\
\frac{L^2}{M^2} &= \frac{(3x-1)^3}{24(2x+1)(x^2+11x-1)},\\
\frac{BL}{M^2} &= -\frac{N_1N_2N_3}{M^3}\\
&= -\frac{1}{24\sqrt{6}}\frac{(3x-1)^{\frac{3}{2}}\sqrt{\{2(18x-1)(9x^3+121x^2+33x+3)\}}}{(2x+1)^{\frac{3}{2}}(x^2+11x-1)},\\
\frac{B}{M} &= \frac{1}{12}\frac{\sqrt{\{2(18x-1)(9x^3+121x^2+33x+3)\}}}{(2x+1)\sqrt{(x^2+11x-1)}},\\
\frac{L}{M} &= -\frac{1}{2\sqrt{6}}\frac{(3x-1)^{\frac{3}{2}}}{\sqrt{\{(2x+1)(x^2+11x-1)\}}},\\
E &= -\frac{2x^2}{LM^2} = \frac{250x^2}{3(2x+1)(x^2+11x-1)},\\
k^2 &= 1-E = \frac{6x^3-181x^2+27x-3}{3(2x+1)(x^2+11x-1)};
\end{aligned} \tag{361}$$

and then

$$\begin{aligned}
Z = &-z^4 - \frac{(3x-1)(x^2+66x-11)}{6(2x+1)(x^2+11x-1)}z^2\\
&-\frac{1}{3\sqrt{6}}\frac{(3x-1)^{\frac{3}{2}}\sqrt{\{2(18x-1)(9x^3+121x^2+33x+3)\}}}{(2x+1)^{\frac{3}{2}}(x^2+11x-1)}z\\
&-\frac{1}{18}\frac{(3x-1)^2(26x^2+21x-21)}{(2x+1)^2(x^2+11x-1)}.
\end{aligned} \tag{362}$$

We must now take

$$\begin{aligned}
H &= \frac{(13x-1)\sqrt{(9x^3+121x^2+33x+3)}}{(3x-1)^{\frac{3}{2}}},\\
K &= \frac{(7x+1)\sqrt{\{2(18x-1)(x^2+11x-1)\}}}{(3x-1)^{\frac{3}{2}}};
\end{aligned} \tag{363}$$

and thence we find

$$
\begin{aligned}
H_2 &= \frac{5}{3}\ \frac{(8x-1)\sqrt{(9x^3+121x^2+33x+3)}}{(2x+1)(3x-1)^{\frac{3}{2}}}, \\
K_2 &= \frac{1}{3}\ \frac{(12x+1)\sqrt{\{2(18x-1)(x^2+11x-1)\}}}{(2x+1)(3x-1)^{\frac{3}{2}}}, \\
H_3 &= \frac{5}{3\sqrt{6}}\ \frac{(7x+1)\sqrt{\{2(18x-1)}}{(2x+1)^{\frac{1}{2}}(3x-1)}, \\
K_3 &= \frac{4}{3\sqrt{6}}\ \frac{\sqrt{(9x^3+121x^2+33x+3)}\sqrt{(x^2+11x-1)}}{(2x+1)^{\frac{3}{2}}(3x-1)}, \\
H_4 &= \frac{10}{9}\ \frac{\sqrt{(3x-1)}\sqrt{(9x^3+121x^2+33x+3)}}{(2x+1)^2}, \\
H_5 &= \frac{1}{9\sqrt{6}}\ \frac{(22x^2+42x+3)\sqrt{\{2(18x-1)\}}}{(2x+1)^{\frac{5}{2}}},
\end{aligned}
\tag{364}
$$

A numerical verification is obtained by taking $x = 2$; this makes

$$
\begin{aligned}
&L = -\tfrac{1}{2},\ \ M = \sqrt{6},\ \ \frac{L}{M} = -\frac{\sqrt{6}}{12},\ \ \frac{B}{M} = \frac{\sqrt{70}}{12}, \\
&E = \tfrac{8}{3},\ \ k^2 = 1 - E = -\tfrac{5}{3};
\end{aligned}
\tag{365}
$$

and then

$$
Z = -z^4 - \frac{5}{6}z^2 - \frac{\sqrt{105}}{9}z - \frac{5}{18}; \tag{366}
$$

and

$$
\begin{aligned}
H &= 5\sqrt{5}, & K &= 3\sqrt{14}; \\
H_2 &= 5\sqrt{5}, & K_2 &= \tfrac{5}{3}\sqrt{14}; \\
H_3 &= \tfrac{5}{3}\sqrt{21}, & K_3 &= \tfrac{2}{9}\sqrt{30}; \\
H_4 &= \tfrac{10}{9}\sqrt{5}, & H_5 &= \tfrac{7}{27}\sqrt{21}.
\end{aligned}
\tag{367}
$$

37. With $n = 7$ and a parameter

$$
v = \tfrac{4}{7}\omega_3, \tag{368}
$$

we take, in (49),

$$
x = -c(1+c)^2, \quad y = -c(1+c); \tag{369}
$$

and now we find

$$
\begin{aligned}
&K \text{ or } H \\
&= \frac{\{c(1+c)^3 + (1+c)^2L - L^2\}\sqrt{\{c^2(1+c)^3 - 2c(1+c)^2L + (1+c-c^2)L^2 + 2L^3\}}}{2L^{\frac{1}{2}} \text{ or } 2(-L)^{\frac{1}{2}}} \\
&H \text{ or } K \\
&= \frac{\{c(1+c)^3 - (1+c)^2L - L^2\}\sqrt{\{c^2(1+c)^3 + 2c(1+c)^2L + (1+c-c^2)L^2 - 2L^3\}}}{2L^{\frac{1}{2}} \text{ or } 2(-L)^{\frac{1}{2}}};
\end{aligned}
\tag{370}
$$

and the solution is made algebraical by taking

$$L = \tfrac{1}{2} P(v) = \frac{3 + 9c + 5c^2}{14}. \tag{371}$$

With $n = 9$ and a parameter

$$v = \tfrac{4}{9} \omega_3, \tag{372}$$

take ("Pseudo-Elliptic Integrals," p. 232)

$$x = p^2 (1-p)(1-p+p^2), \quad y = p^2(1-p); \tag{373}$$

and now

K or $H =$

$$\frac{\{p^6(1-p)(1-p+p^2) - p^4(1-p+p^2)L - p^2(1-2p)L^2 + L^3\} \sqrt{\{p^4(1-p)(1-p+p^2) + 2p^2(1-p)(1-p+p^2)L + (1+0+p^2-p^3)^2 L + 2L^3\}}}{2L^{\frac{3}{2}} \text{ or } 2(-L)^{\frac{3}{2}}},$$

H or $K =$

$$\frac{\{p^4(1-p)(1-p+p^2) + p^4(1-p+p^2)L - p^2(1-2p)L^2 - L^3\} \sqrt{\{p^4(1-p)(1-p+p^2) - 2p^2(1-p)(1-p+p^2)L + (1+0-p^2-p^3)L^2 - 2L^3\}}}{2L^{\frac{3}{2}} \text{ or } 2(-L)^{\frac{3}{2}}}, \tag{375}$$

and so on for the higher values of n, but the complexity increases very rapidly.

38. Making use of the theorem in §30 that $I = K$ or H and $J = H$ or K, according as the body is prolate or oblate, in the associated pseudo-elliptic expressions of ϖ in (184), we can now write down the corresponding formulas, and determine the remaining coefficients by means of the differential relation

$$\frac{d(\varpi - pt)}{dz} = \frac{-\dfrac{L + \frac{1}{2}P}{M} z^2 + 2\dfrac{B}{M} z - \dfrac{L - \frac{1}{2}P}{M}(1 + aE)}{(1 + aE - z^2)\sqrt{Z}}. \tag{192}$$

Thus with the parameter $v = \omega_1 + \frac{1}{2}\omega_3$,

$$\begin{aligned} \varpi - pt &= \tfrac{1}{2} \cos^{-1} \frac{Jz + I_1}{1 + E - z^2} \sqrt{(z_0 - z \,.\, z_3 - z)} \\ &= \tfrac{1}{2} \sin^{-1} \frac{Jz + J_1}{1 + E - z^2} \sqrt{(z - z_2 \,.\, z - z_1)}; \end{aligned} \tag{376}$$

with

$$\begin{aligned} I^2 = K^2 &= \frac{L^2 + N_1 N_2}{2L^2}, \\ J^2 = H^2 &= \frac{L^2 - N_1 N_2}{2L^2}; \end{aligned} \tag{377}$$

and then

$$\frac{I_1}{J} = \frac{N_1N_2 + (L+1)^2}{M}$$
$$\frac{J_1}{I} = \frac{N_1N_2 - (L+1)^2}{M}. \tag{378}$$

The relation (376) may also be written

$$\left\{\frac{F}{An}\rho e^{i(\varpi - pt)}\right\}^2 = (Iz + I_1)\sqrt{(z_0 - z \,.\, z - z_3)} + i(Jz + J_1)\sqrt{(z - z_2 \,.\, z - z_1)}. \tag{379}$$

With a parameter $v = \frac{1}{2}\omega_3$,

$$\varpi - pt = \tfrac{1}{2}\cos^{-1}\frac{Iz + I_1}{z^2 - 1 + E}\sqrt{(z - z_1 \,.\, z - z_2)}$$
$$= \tfrac{1}{2}\sin^{-1}\frac{Jz + J_1}{z^2 - 1 + E}\sqrt{(z_3 - z \,.\, z - z_0)}, \tag{380}$$

or

$$\left\{\frac{F}{An}\rho e^{i(\varpi - pt)}\right\}^2 = (Iz + I_1)\sqrt{(z - z_1 \,.\, z - z_2)} + i(Jz + J_1)\sqrt{(z_3 - z \,.\, z - z_0)}, \tag{381}$$

with

$$I^2 = H^2 = \frac{N_1N_2 - L^2}{2L^2},$$
$$J^2 = K^2 = \frac{N_1N_2 + L^2}{2L^2}; \tag{382}$$

and then

$$\frac{I_1}{J} = \frac{(L + P)^2 + N_1N_2}{PM},$$
$$\frac{J_1}{I} = \frac{(L + P)^2 - N_1N_2}{LM}. \tag{383}$$

So also with a parameter

$$v = \omega_1 + \tfrac{2}{3}\omega_3, \text{ or } \omega_1 + \tfrac{1}{3}\omega_3, \text{ or } \tfrac{1}{3}\omega_3, \tag{270}$$

the expressions take the form

$$\left\{\frac{F}{An}\rho e^{i(\varpi - pt)}\right\}^3$$
$$= (Iz^2 + I_1z + I_2)\sqrt{(z_0 - z \,.\, z - z_1)} + i(Jz^2 + J_1z + J_2)\sqrt{(z_3 - z \,.\, z - z_2)}, \tag{384}$$

or
$$= (Iz^2 + I_1z + I_2)\sqrt{(z_3 - z \,.\, z - z_1)} + i(Jz^2 + J_1z + J_2)\sqrt{(z_0 - z \,.\, z - z_2)}, \tag{385}$$

or
$$= (Iz^2 + I_1z + I_2)\sqrt{(z - z_1 \,.\, z - z_2)} + i(Jz^2 + J_1z + J_2)\sqrt{(z_3 - z \,.\, z - z_0)}. \tag{386}$$

Thus, in completion of the preceding special cases of §§28, 29, we find for a parameter $v = \omega_1 + \frac{1}{2}\omega_3$, and, in addition, with

$$B = 0,\ N_1 = 0,\ L = \sqrt{(1+c)},$$
$$M = \sqrt{c+c^2} - \sqrt{c};\ N_2 = \sqrt{(1+2c)},\ N_3 = 1+c,$$
$$1 + E = \left\{\frac{\sqrt{(1+c)}+1}{\sqrt{(1+c)}-1}\right\}^2 = k^2,$$

$$\frac{N_2^2 - N_3^2}{M^2} = z_0 z_3 = z_1 z_2 = \frac{\sqrt{(1+c)}+1}{\sqrt{(1+c)}-1} = k;$$
$$\varpi - pt = \tfrac{1}{2}\cos^{-1}\frac{z + k^{\frac{3}{2}}}{\sqrt{2(k^2 - z^2)}}\sqrt{\left(z^2 - 2\frac{N_3}{M}z + k\right)}$$
$$= \tfrac{1}{2}\sin^{-1}\frac{z - k^{\frac{3}{2}}}{\sqrt{2(k^2 - z^2)}}\sqrt{\left(z^2 + 2\frac{N_3}{M}z + k\right)}; \tag{386a}$$

with a corresponding

$$\psi - pt = \tfrac{1}{2}\cos^{-1}\frac{z + \dfrac{\sqrt{(1+c)}-1}{\sqrt{c}}}{\sqrt{2(1-z^2)}}\sqrt{\left(z^2 - 2\frac{N_3}{M}z + k\right)}$$

$$= \tfrac{1}{2}\sin^{-1}\frac{z - \dfrac{\sqrt{(1+c)}-1}{\sqrt{c}}}{\sqrt{2(1-z^2)}}\sqrt{\left(z^2 + 2\frac{N_3}{M}z + k\right)}. \tag{386b}$$

For the parameter $v = \frac{1}{2}\omega_3$, with

$$B = 0,\ N_3 = 0,\ k^2 = 1 - E = \{\sqrt{(1+c)} - \sqrt{c}\}^4,\ L = -\sqrt{(c+c^2)},$$
$$\varpi - pt = \tfrac{1}{2}\cos^{-1}\frac{z}{z^2 - \{\sqrt{(1+c)} - \sqrt{c}\}^4}\sqrt{\left[z^2 - \frac{\{\sqrt{(1+c)}-\sqrt{c}\}^2}{1+2c}\right]}$$
$$= \tfrac{1}{2}\sin^{-1}\frac{\dfrac{\{\sqrt{(1+c)}-\sqrt{c}\}^3}{\sqrt{(1+2c)}}}{z^2 - \{\sqrt{(1+c)}-\sqrt{c}\}^4}\sqrt{[(1+2c)\{\sqrt{(1+c)}-\sqrt{c}\}^2 - z^2]}. \tag{386c}$$

In the numerical case of the parameter $v = \omega_1 + \frac{1}{3}\omega_3$ in §29, we find

$$\varpi - nt = \tfrac{1}{3}\cos^{-1}\frac{-z^3 - 8\sqrt{7}z + 131}{(13 - z^2)^{\frac{3}{2}}}\sqrt{(-z^2 - 2\sqrt{7}z + 5)}$$
$$= \tfrac{1}{3}\sin^{-1}\frac{\sqrt{7}z^3 - 2z - 57\sqrt{7}}{(13 - z^2)^{\frac{3}{2}}}\sqrt{(-z^2 + 2\sqrt{7}z - 3)}. \tag{386d}$$

Or, in the algebraical case, with

$$B = 0,\ N_1 = 0,\ L = (1-c)\sqrt{(1-2c)},$$
$$M = \sqrt{(2c - c^2)}\{1 - c - \sqrt{(1-2c)}\},\ N_2 = \sqrt{(1-2c)},\ N_3 = (1-c)\sqrt{(1-c^2)},$$

we find

$$H = K = \frac{1}{\sqrt 2}$$

and

$$k^2 = 1 + E = \left\{\frac{1+\sqrt{(1-2c)}}{1-\sqrt{(1-2c)}}\right\}^4,$$

and now the result can be written

$$\varpi - pt = \tfrac13 \cos^{-1} \frac{z^2 + I_1 z + I_2}{\sqrt 2\,(k^2 - z^2)^{\frac32}} \sqrt{\left(-z^2 + 2\,\frac{N_2}{M}\,z + k\right)}$$

$$= \tfrac13 \sin^{-1} \frac{z^2 - I_1 z + I_2}{\sqrt 2\,(k^2 - z^2)^{\frac32}} \sqrt{\left(-z^2 - 2\,\frac{N_2}{M}\,z + k\right)}, \qquad (386e)$$

where
$$I_1 = \frac{N_2 - 3L - (2-c)(1-2c)}{M}, \quad I_2 = -k^{\frac32},$$

and then
$$\psi - pt = \tfrac13 \cos^{-1} \frac{z^2 + H_1 z + H_2}{\sqrt 2\,(1 - z^2)^{\frac32}} \sqrt{\left(-z^2 - 2\,\frac{N_2}{M}\,z + k\right)}$$

$$= \tfrac13 \sin^{-1} \frac{z^2 - H_1 z + H_2}{\sqrt 2\,(1 - z^2)^{\frac32}} \sqrt{\left(-z^2 + 2\,\frac{N_2}{M}\,z + k\right)}, \qquad (386f)$$

where
$$H_1 = \frac{\sqrt{(1-2c)}}{\sqrt{(2c-c^2)}} \{1 - \sqrt{(1-2c)}\}; \quad H_2 = -\frac{1-\sqrt{(1-2c)}}{1+\sqrt{(1-2c)}}.$$

With a parameter $v = \omega_1 + \frac23 \omega_3$, and, as in (284$b$),

$$B = 0,\ N_1 = 0;\ k^2 = 1 + E = \left\{\frac{\sqrt{(1-c^2)} + \sqrt{(1-2c)}}{\sqrt{(1-c^2)} - \sqrt{(1-2c)}}\right\},$$

$$\varpi - pt = \tfrac13 \cos^{-1} \frac{z^2 + I_2}{(k^2 - z^2)^{\frac32}} \sqrt{\left\{\left(\frac{N_2 - N_3}{M}\right)^2 - z^2\right\}}$$

$$= \tfrac13 \sin^{-1} \frac{J_1 z}{(k^2 - z^2)^{\frac32}} \sqrt{\left\{\left(\frac{N_2 + N_3}{M}\right)^2 - z^2\right\}}, \qquad (386g)$$

where
$$J_1 = \frac{3L + (1+c)^{\frac12}(1-2c)}{M}, \quad I_2 = -k^3\,\frac{M}{N_2 - N_3}.$$

So also with a parameter $v = \frac13 \omega_3$, and $B = 0$, $N_3 = 0$,

$$\varpi - pt = \tfrac13 \cos^{-1} \frac{I_1 z}{(z^2 - k^2)^{\frac32}} \sqrt{\left\{-z^2 + \left(\frac{N_1 + N_2}{M}\right)^2\right\}}$$

$$= \tfrac13 \sin^{-1} \frac{z^2 + J_2}{(z^2 - k^2)^{\frac32}} \sqrt{\left\{z^2 - \left(\frac{N_1 - N_2}{M}\right)^2\right\}}, \qquad (386h)$$

where
$$k^2 = 1 - E = \left\{\frac{\sqrt{(1-c^2)} - \sqrt{(2c-c^2)}}{\sqrt{(1-c^2)} + \sqrt{(2c-c^2)}}\right\}^2 \text{ etc.}$$

The expression for ϖ in the algebraical case, when the parameter $v = \frac{4}{5}\omega_3$, has been given already in (347) and (349).

So also, corresponding to the algebraical case in (358), we shall find

$$\varpi = \tfrac{1}{5}\cos^{-1}\frac{Iz^5 + I_1z^4 + I_2z^3 + I_3z^2 + I_4 + I_5}{(z^2 - 1 + E)^{\frac{5}{2}}}$$
$$= \tfrac{1}{5}\sin^{-1}\frac{Jz^3 + J_1z^2 + J_2z + J_3}{(z^2 - 1 + E)^{\frac{5}{2}}}\sqrt{Z},$$

or

$$\left(\frac{F}{An}\rho e^{\varpi i}\right)^5 = \left\{\frac{F}{An}(\alpha + \beta i)\right\}^5$$
$$= (Iz^5 + I_1z^4 + \dots + I_5) + i\,(Jz^3 + J_1z^2 + J_2z + J_3)\sqrt{Z}, \qquad (387)$$

where

$$I = K = \frac{(7x + 1)\sqrt{\{2\,(18x - 1)(x^2 + 11x - 1)\}}}{(3x - 1)^{\frac{5}{2}}},$$

$$J = H = \frac{(13x - 1)\sqrt{(9x^3 + 121x^2 + 33x + 3)}}{(3x - 1)^{\frac{5}{2}}}; \qquad (388)$$

and then

$$I_1 = 10\frac{L}{M}J = -\frac{5}{\sqrt{6}}\,\frac{(13x - 1)\sqrt{(9x^3 + 121x^2 + 33x + 3)}}{(3x - 1)\sqrt{\{(2x + 1)(x^2 + 11x - 1)\}}},$$

$$J_1 = 10\frac{L}{M}I = -\frac{5}{\sqrt{6}}\,\frac{(7x + 1)\sqrt{\{2\,(18x - 1)\}}}{(3x - 1)\sqrt{(2x + 1)}},$$

$$I_2 = \frac{3}{5}\,\frac{(31x^2 + x - 1)\sqrt{\{2\,(18x - 1)\}}}{(3x - 1)^{\frac{3}{2}}\sqrt{(x^2 + 11x - 1)}},$$

$$J_2 = \frac{1}{3}\,\frac{(18x - 1)(31x^2 + x - 1)\sqrt{(9x^3 + 121x^2 + 33x + 3)}}{(2x + 1)(x^2 + 11x - 1)(3x - 1)^{\frac{3}{2}}},$$

$$I_3 = -\frac{10}{3\sqrt{6}}\,\frac{(122x^4 + 309x^3 - 357x^2 + 66x - 3)\sqrt{(9x^3 + 121x^2 + 33x + 3)}}{(2x + 1)^{\frac{3}{2}}(x^2 + 11x - 1)^{\frac{3}{2}}(3x - 1)};$$

$$J_3 = -\frac{1}{3\sqrt{6}}\,\frac{(31x^2 + x - 1)(26x^2 + 21x - 21)\sqrt{\{2\,(18x - 1)\}}}{(3x - 1)(x^2 + 11x - 1)(2x + 1)^{\frac{3}{2}}},$$

$$I_4 = 0,$$

$$I_5 = -\tfrac{2}{5}I_3(1 - E) - 2\frac{B}{M}J_3 \qquad (389)$$

$$= \frac{1}{9\sqrt{6}}\,\frac{\sqrt{(9x^3 + 121x^2 + 33x + 3)}}{(2x + 1)^{\frac{5}{2}}(x^2 + 11x - 1)^{\frac{5}{2}}}(5812x^6 + 31956x^5 - 29655x^4$$
$$+ 37830x^3 - 13495x^2 + 1596x - 57). \qquad (389)$$

The special numerical case of $x = 2$ makes

$$I = 3\sqrt{14},\ I_1 = -\frac{25\sqrt{30}}{5},\ I_2 = \frac{25\sqrt{14}}{3},\ I_3 = -\frac{25\sqrt{30}}{9},\ I_4 = 0,\ I_5 = \frac{25\sqrt{30}}{18},$$

$$J = 5\sqrt{5},\ J_1 = -5\sqrt{21},\ J_2 = \frac{35\sqrt{5}}{3},\ J_3 = -\frac{25\sqrt{21}}{9},$$

with
$$\frac{L}{M} = -\frac{\sqrt{6}}{12},\quad \frac{B}{M} = \frac{\sqrt{70}}{12},$$

and
$$Z = -z^4 - \frac{5}{6}z^3 - \frac{\sqrt{105}}{9}z - \frac{5}{18};$$

and these numbers will be found to verify.

39. In some cases a factor of Z, say $z - z_0$, can be obtained; and with

$$Z = az^4 + 4bz^3 + 6cz^2 + 4dz + e, \tag{27}$$

we can take Weierstrass's formula

$$\wp u = \tfrac{1}{3}(az_0^2 + 2bz_0 + c) + \frac{az_0^3 + 3bz_0^2 + 3cz_0 + d}{z - z_0}, \tag{390}$$

and now we put

$$\frac{\sigma - s}{M^2} = \wp v - \wp u, \tag{391}$$

where $\wp v$ has the expression given in (86).

The more general form of T, involving circulation, etc., may now be employed, without introducing additional complication.

It is convenient to put

$$1 + aE = k^2, \tag{392}$$

so that

$$Z = a(z^2 - 1)(z^2 - k^2) - 4\left(\frac{Lz - B}{M}\right)^2; \tag{393}$$

thus $z \mp 1$ is a factor of Z if $B = \pm L$, and $z \mp k$ is a factor if $B = \pm Lk$.

The case of $B = L$ may be supposed to be produced when the body, projected originally as a perfectly centred projectile from a rifled gun, has struck an obstacle, thus setting up gyrations in which the axis OC periodically passes through Oz, as in the corresponding *rosette* curves described by the axis of a top discussed in Professor Klein's paper on "The Stability of a Sleeping Top," in the Bulletin of the American Mathematical Society, Jan. 1897.

40. Now, with $B = L$, and $z - 1$ a factor of Z in equation (390),

$$\wp u = \tfrac{1}{2}(a + c) + \frac{a + 3c + d}{z - 1}, \tag{394}$$

and

$$c = -\tfrac{1}{3}a - \tfrac{1}{6}E - \tfrac{2}{3}\frac{L^2}{M^2}, \tag{395}$$

$$d = 2\frac{BL}{M^2} = 2\frac{L^2}{M^2}; \tag{396}$$

so that

$$\wp v - \wp u = \tfrac{1}{4}E + \frac{\tfrac{1}{2}E}{z - 1}, \tag{397}$$

$$\frac{\sigma - s}{M^2} = \tfrac{1}{4}E\frac{z + 1}{z - 1}; \tag{398}$$

or from (65),

$$\frac{\sigma - s}{\sqrt{-\Sigma}} = -\frac{a}{2L}\,\frac{1 + z}{1 - z}, \tag{399}$$

$$\frac{\sigma - s_a}{\sqrt{-\Sigma}} = -\frac{a}{2L}\,\frac{1 + z_a}{1 - z_a}, \tag{400}$$

$$\frac{s - s_a}{\sqrt{-\Sigma}} = -\frac{a}{L}\,\frac{z - z_a}{(1 - z_a)(1 - z)}, \tag{401}$$

or

$$\frac{s - s_a}{M^2} = \tfrac{1}{2}E\frac{z - z_a}{(1 - z_a)(1 - z)}. \tag{402}$$

But from (393),

$$a(z - z_\alpha)(z - z_\beta)(z - z_\gamma) = a(z + 1)(z^2 - k^2) - 4\frac{L^2}{M^2}(z - 1), \tag{403}$$

so that

$$-a(1 - z_\alpha)(1 - z_\beta)(1 - z_\gamma) = 2a(1 - k^2) = -2E, \tag{404}$$

and

$$\frac{\tfrac{1}{4}S}{M^6} = \tfrac{1}{8}E^3\frac{Z}{-a(1 - z_\alpha)(1 - z_\beta)(1 - z_\gamma)(z - 1)^4}$$

$$= \tfrac{1}{16}E^2\frac{Z}{(z - 1)^4},$$

$$\frac{\sqrt{S}}{M^3} = \tfrac{1}{2}E\frac{\sqrt{Z}}{(z - 1)^4}, \tag{405}$$

and

$$\frac{ds}{M^2} = \tfrac{1}{2}E\frac{dz}{(z - 1)^2}, \tag{406}$$

so that, as in (48),

$$M\frac{ds}{\sqrt{S}} = \frac{dz}{\sqrt{Z}}. \tag{407}$$

With $L = B$ in equation (59),

$$\frac{L}{M} = \frac{B}{M} = a\frac{N_1 N_2 N_3}{LM^2},$$

or

$$L^2 M = a N_1 N_2 N_3, \tag{408}$$

so that, from (63),

$$aL^4 M^2 = L^4 \left(L^2 + 3M^2 \wp v - a\frac{\sqrt{-\Sigma}}{L}\right)$$

$$= aN_1^2 N_2^2 N_3^2 = (L^2 + \sigma - s_1)(L^2 + \sigma - s_2)(L^2 + \sigma - s_3)$$

$$= L^6 + 3L^4 M^2 \wp v + \tfrac{1}{2} L^2 M^4 \wp'' v + \tfrac{1}{4}\Sigma, \tag{409}$$

or

$$aL^3\sqrt{-\Sigma} + \tfrac{1}{2} L^2 M^4 \wp'' v + \tfrac{1}{4}\Sigma = 0, \tag{410}$$

so that L must be determined from this cubic equation in a very similar manner to that required in the Spherical Pendulum, as discussed in the Proceedings of the London Math. Society, vol. XXVII, p. 607.

Having determined L, or, in a pseudo-elliptic case, having expressed L, σ, $\sqrt{-\Sigma}$, $\wp v$, $\wp'' v$ and P in terms of a single parameter,

$$d\psi = 2\frac{L}{M}\frac{1}{1+z}\frac{dz}{\sqrt{Z}}$$

$$= \frac{2L}{1+z}\frac{ds}{\sqrt{S}}, \tag{411}$$

and

$$\frac{z+1}{z-1} = \frac{\sigma - s}{\frac{1}{4}EM^2} = \frac{L}{\frac{1}{2}a\sqrt{-\Sigma}}(\sigma - s),$$

$$z + 1 = \frac{2L(\sigma - s)}{L(\sigma - s) - \frac{1}{2}a\sqrt{-\Sigma}}, \tag{412}$$

so that

$$d\psi = \frac{L(\sigma - s) - \frac{1}{2}a\sqrt{-\Sigma}}{\sigma - s}\frac{ds}{\sqrt{S}}$$

$$= (L - \tfrac{1}{2}P)\frac{ds}{\sqrt{S}} + \tfrac{1}{2}\frac{P(\sigma \sim s) - \sqrt{-\Sigma}}{\sigma \sim s}\frac{ds}{\sqrt{S}}, \tag{413}$$

$$\psi = \frac{L - \frac{1}{2}P}{M}nt + I(v), \tag{414}$$

and similarly for ϖ, which can be made to depend upon the same integral $I(v)$.

The case of $B = -L$ is the same as the preceding, with z changed into $-z$, so that this case does not require separate treatment.

A reference to equation (14) shows that, with $B = \pm L$,

$$\frac{d\psi_1}{dt} = 0, \quad \text{or} \quad \frac{d\psi_2}{dt} = 0,$$

so that
$$\frac{d\phi}{dt} = \left(1 - \frac{C}{A}\right)n \pm \frac{d\psi}{dt}, \tag{415}$$

and ϕ is now pseudo-elliptic with ψ, as also $u + vi$ and $p + qi$, on reference to (179) and (180).

41. A numerical case will serve to elucidate the preceding theory.

Take $v = \frac{4}{5}\omega_3$, and in (410), $a = -1$, $y = x = -\frac{1}{2}$, when
$$\sqrt{-\Sigma} = x^2 = \tfrac{1}{4}, \quad M^4\wp''v = x^2 - x^3 = \tfrac{3}{8}, \tag{416}$$

and then
$$-\tfrac{1}{4}L^3 + \tfrac{3}{16}L^2 - \tfrac{1}{64} = 0, \tag{417}$$

which is satisfied by $L = -\frac{1}{4}$; and now, in (353),
$$H = 2\sqrt{2}, \quad K = -3. \tag{418}$$

Also in (329),
$$M^2 = \tfrac{1}{4}(1+x)^2 - 2x - L^2 - \frac{x^2}{L} = 2, \tag{419}$$
$$LM^2 = -\tfrac{1}{2}, \quad \frac{L}{M} = -\tfrac{1}{8}\sqrt{2}; \tag{420}$$

and in (65) and (59),
$$E = -\frac{2x^3}{LM^3} = 1, \tag{421}$$
$$BLM = -N_1N_2N_3 = -\frac{1}{8\sqrt{2}}, \tag{422}$$
$$\frac{B}{M} = \frac{1}{4\sqrt{2}} = -\frac{L}{M}, \tag{423}$$

and
$$\begin{aligned} Z &= -z^2(z^2-1) - \tfrac{1}{8}(z+1)^2 \\ &= (z+1)(-z^3 + z^2 - \tfrac{1}{8}z - \tfrac{1}{8}). \end{aligned} \tag{424}$$

We therefore take, from (398) and (405),
$$s = \tfrac{1}{2}\frac{1-z}{1+z}, \tag{425}$$
$$s + x = \frac{-z}{1+z}, \tag{426}$$
$$S = \frac{2Z}{(1+z)^4}; \tag{427}$$

and now in (147),
$$\frac{p}{n} = -\tfrac{1}{4}\sqrt{2}; \tag{428}$$

so that

$$\frac{d\psi}{dz} = \tfrac{1}{4}\sqrt{2}\,\frac{1}{(1-z)\sqrt{Z}}, \tag{429}$$

$$\frac{d(\psi - pt)}{dz} = \tfrac{1}{4}\sqrt{2}\,\frac{-z}{(1-z)\sqrt{Z}}. \tag{430}$$

Taking the pseudo-elliptic integral (Proc. L. M. S. XXV, p. 214)

$$\begin{aligned} I(\tfrac{4}{5}\omega_3) &= \tfrac{1}{5}\int \frac{\frac{1}{5}(1-3x)s - x^2}{s\sqrt{S}}\,ds \\ &= \tfrac{1}{5}\cos^{-1}\frac{(1-3x)s^2 - (2x^2 - x^3)s + x^4}{2s^{\frac{5}{2}}} \\ &= \tfrac{1}{5}\sin^{-1}\frac{s - x^2}{2s^{\frac{5}{2}}}\sqrt{S}, \end{aligned} \tag{431}$$

with $y = x = -\frac{1}{2}$, and making use of (425), we find that

$$\begin{aligned} \psi - pt &= \tfrac{2}{5}\cos^{-1}\frac{2\sqrt{2}\,z^2 - \frac{9}{4}\sqrt{2}\,z + \frac{3}{4}\sqrt{2}}{(1-z)^{\frac{5}{2}}}\sqrt{(z+1)} \\ &= \tfrac{2}{5}\sin^{-1}\frac{-3z+1}{(1-z)^{\frac{5}{2}}}\sqrt{(-z^3 + z^2 - \tfrac{1}{8}z - \tfrac{1}{8})}, \end{aligned} \tag{432}$$

and this, on differentiation, will be found to satisfy (430).

As for equations (165) and (169), they become

$$\frac{F\rho}{An} = -z, \tag{433}$$

and

$$\frac{d\varpi}{dz} = \tfrac{1}{4}\sqrt{2}\,\frac{1+z}{-z\sqrt{Z}}, \tag{434}$$

or

$$\frac{d(\varpi - pt)}{dz} = \tfrac{1}{4}\sqrt{2}\,\frac{1}{-z\sqrt{Z}}; \tag{435}$$

the integral of which, derived from the pseudo-elliptic integral (Proc. L. M. S. XXV, p. 213),

$$\begin{aligned} I(\tfrac{2}{5}\omega_3) &= \tfrac{1}{5}\int \frac{\frac{1}{5}(3+x)(s+x) - x}{(s+x)\sqrt{S}} \\ &= \tfrac{1}{5}\cos^{-1}\frac{(3+x)s^2 - (1 - 4x - 2x^2)s + x^2 + x^3}{2(s+x)^{\frac{5}{2}}} \\ &= \tfrac{1}{5}\sin^{-1}\frac{s - 1 + x}{2(s+x)^{\frac{5}{2}}}\sqrt{S}, \end{aligned} \tag{436}$$

by making use of (426), becomes

$$\varpi - pt = \tfrac{2}{3}\cos^{-1}\frac{z^2 - \frac{1}{2}z - \frac{1}{4}}{(-z)^{\frac{3}{2}}}\sqrt{(z+1)}$$

$$= \tfrac{2}{3}\sin^{-1}\frac{\sqrt{2}(z+\frac{1}{2})}{(-z)^{\frac{3}{2}}}\sqrt{(-z^3+z^2-\tfrac{1}{8}z-\tfrac{1}{8})}, \tag{437}$$

which will be found to satisfy (435).

Equations (432) and (437) can also be written

$$\{\tan^2\tfrac{1}{2}\theta e^{(\psi-pt)i}\}^{\frac{3}{2}} = \sqrt{2}\,\frac{2z^2-\frac{9}{4}z+\frac{3}{4}}{(z+1)^2} + i\,\frac{-3z+1}{z+1}\sqrt{\frac{-z^3+z^2-\frac{1}{8}z-\frac{1}{8}}{(z+1)^3}}, \tag{438}$$

$$\left\{\frac{F}{An}\rho e^{(\varpi-pt)i}\right\}^{\frac{3}{2}} = (z^2-\tfrac{1}{2}z-\tfrac{1}{4})\sqrt{(z+1)} + i\sqrt{2}(z+\tfrac{1}{2})\sqrt{(-z^3+z^2-\tfrac{1}{8}z-\tfrac{1}{8})}. \tag{439}$$

Now, in (179),

$$P(u+vi) = -F\sin\theta e^{-\left(1-\frac{C}{A}\right)rti+pti}e^{(\psi-pt)i}, \tag{440}$$

while

$$\{\sin\theta e^{(\psi-pt)i}\}^{\frac{3}{2}} = \sqrt{2}\,(2z^2-\tfrac{9}{4}z+\tfrac{3}{4})(z+1)^3 + i(-3z+1)(z+1)^2\sqrt{Z}. \tag{441}$$

But, while keeping Cr unchanged, C and r may be varied so as to make

$$\left(1-\frac{C}{A}\right)r = p, \tag{442}$$

and now

$$\left\{\frac{P}{F}(u+vi)\right\}^{\frac{3}{2}} = -\sqrt{2}\,(2z^2-\frac{9}{4}z+\tfrac{3}{4})(z+1)^3 + i(3z-1)(z+1^2)\sqrt{Z}, \tag{443}$$

with a similar expression for $p + qi$ obtained from (180).

42. With
$$B - Lk = 0 \tag{444}$$
and
$$1 + aE = \frac{B^2}{L^2}, \tag{445}$$

$$Z = \left(z - \frac{B}{L}\right)\left\{a(z^2-1)\left(z+\frac{B}{L}\right) - 4\frac{L^2}{M^2}\left(z-\frac{B}{L}\right)\right\}. \tag{446}$$

A similar calculation shows that

$$\frac{\sigma - s}{M^2} = \tfrac{1}{4}a\left(\frac{B^2}{L^2}-1\right)\frac{z+\frac{B}{L}}{z-\frac{B}{L}}, \tag{447}$$

and
$$aL^3\sqrt{-\Sigma} - \tfrac{1}{2}L^2\wp''v - \tfrac{1}{4}\Sigma = 0, \tag{448}$$

the same as (410), when L is changed into $-L$.

With $z-\dfrac{B}{L}$ a factor of Z, $\dfrac{d\varpi}{dz}=0$ when $z=\dfrac{B}{L}$, and the curve $(\alpha,\ \beta)$ has a series of cusps.

But with $z-\dfrac{L}{B}$ a factor of Z, $\dfrac{d\psi}{dz}=0$ when $z=\dfrac{L}{B}$, and the cone described by OC round OC has a series of cuspidal edges; and in this case we find

$$\wp u-\wp w=\tfrac{1}{4}a\left(\frac{L^2}{B^2}-1\right)\frac{z+\dfrac{L}{B}}{z-\dfrac{L}{B}}. \tag{449}$$

43. The integrals are *non-elliptic* when the quartic Z has a pair of equal roots; this will be the case if the body is projected in the direction of its axis OC, with a rotation about OC, as if fired from a rifled gun and perfectly centred.

Denoting the velocity of projection by V, and the angular velocity by r; then in the preceding notation,

$F=RV;\quad G=CrRV=CrF,$ or $B=L;\quad E=0;$

$$2T=RV^2+Cr^2=\frac{F^2}{R}+Cr^2; \tag{450}$$

$$\begin{aligned} Z&=a(z^2-1)^2-4\frac{L^2}{M^2}(z-1)^2\\ &=(z-1)^2\left\{a(z+1)^2-4\frac{L^2}{M^2}\right\}, \end{aligned} \tag{451}$$

$$\frac{ndt}{dz}=\frac{1}{(1-z)\sqrt{\left\{a(1+z)^2-4\dfrac{L^2}{M^2}\right\}}}, \tag{452}$$

$$\frac{d\psi}{dz}=2\frac{L}{M}\frac{1}{(1-z^2)\sqrt{\left\{a(1+z)^2-4\dfrac{L^2}{M^2}\right\}}}. \tag{453}$$

If the body is oblate and $a=-1$, the only solution is $z=1$, so that this motion is stable; but, at the same time, putting

$$1-z=y^2, \tag{454}$$

then
$$\frac{dy^2}{dt^2}=n^2y^2\left(-1-\frac{L^2}{M^2}+y^2-\tfrac{1}{4}y^4\right), \tag{455}$$

so that
$$\frac{d^2y}{dt^2} = -n^2\left(1 + \frac{L^2}{M^2}\right)y + \ldots .$$
$$\approx -n^2\left(1 + \frac{L^2}{M^2}\right)y, \tag{456}$$

when y is small; so that the number of complete oscillations in one second is
$$\frac{n}{2\pi}\sqrt{\left(1 + \frac{L^2}{M^2}\right)}. \tag{457}$$

But in a prolate body, with $a = +1$, then (i) with $\frac{L}{M} > 1$,
$$n\frac{dt}{dz} = \frac{1}{(1-z)\sqrt{\left\{(1+z)^2 - \frac{L^2}{M^2}\right\}}}; \tag{458}$$

and integrating,
$$nt = \frac{1}{\sqrt{\left(\frac{L^2}{M^2} - 1\right)}}\cos^{-1}\sqrt{\frac{\left(\frac{L}{M}+1\right)\left(1+z-2\frac{L}{M}\right)}{2\frac{L}{M}(1-z)}}$$
$$= \frac{1}{\sqrt{\left(\frac{L^2}{M^2} - 1\right)}}\sin^{-1}\sqrt{\frac{\left(\frac{L}{M}-1\right)\left(1+z+2\frac{L}{M}\right)}{2\frac{L}{M}(1-z)}}, \tag{459}$$

but this does not represent any real motion of the axis, since z oscillates between ± 1.

The only solution is therefore $z = 1$; and to find the period of a small oscillation, putting
$$1 - z = y^2 \tag{454}$$
as before,
$$\frac{dy^2}{dt^2} = n^2y^2\left(1 - \frac{L^2}{M^2} - y^2 + \tfrac{1}{4}y^4\right), \tag{455}$$
$$\frac{d^2y}{dt^2} = -n^2\left(\frac{L^2}{M^2} - 1\right)y + \ldots, \tag{456}$$

and the axis thus makes
$$\frac{n}{2\pi}\sqrt{\left(\frac{L^2}{M^2} - 1\right)} \tag{457}$$

complete oscillations per second.

Thus to secure the stability of this elongated projectile in its flight we must make

$$\frac{L}{M}=\frac{Cr}{2An}>1. \tag{458}$$

44. But (ii) with $\frac{L}{M}<1$, integrating

$$nt=\frac{1}{\sqrt{\left(1-\frac{L^2}{M^2}\right)}}\operatorname{ch}^{-1}\sqrt{\frac{\left(1-\frac{L}{M}\right)\left(1+z+2\frac{L}{M}\right)}{2\frac{L}{M}(1-z)}}$$

$$=\frac{1}{\sqrt{\left(1-\frac{L^2}{M^2}\right)}}\operatorname{sh}^{-1}\sqrt{\frac{\left(1+\frac{L}{M}\right)\left(1+z-2\frac{L}{M}\right)}{2\frac{L}{M}(1-z)}}, \tag{459}$$

so that the motion of the axis is unstable, and, after an infinite time, is given by the equation (459) above.

Putting

$$\frac{L}{M}=\cos\alpha, \tag{460}$$

$$\operatorname{ch}^2(nt\sin\alpha)=\frac{(1-\cos\alpha)(1+z+2\cos\alpha)}{2\cos\alpha(1-z)}, \tag{461}$$

$$\operatorname{sh}^2(nt\sin\alpha)=\frac{(1+\cos\alpha)(1+z-2\cos\alpha)}{2\cos\alpha(1-z)}, \tag{462}$$

$$\operatorname{ch}(2nt\sin\alpha)=\frac{z-\cos 2\alpha}{\cos\alpha(1+z)}, \tag{463}$$

$$z=\frac{\cos 2\alpha+\cos\alpha\operatorname{ch}(2nt\sin\alpha)}{1+\cos\alpha\operatorname{ch}(2nt\sin\alpha)}, \tag{464}$$

$$1+z=2\cos\alpha\frac{\cos\alpha+\operatorname{ch}(2nt\sin\alpha)}{1+\cos\alpha\operatorname{ch}(2nt\sin\alpha)}; \tag{465}$$

$$\frac{d\psi}{dt}=\frac{2n\cos\alpha}{1+z}$$

$$=n\frac{1+\cos\alpha\operatorname{ch}(2nt\sin\alpha)}{\cos\alpha+\operatorname{ch}(2nt\sin\alpha)}$$

$$=n\cos\alpha+\frac{n\sin^2\alpha}{\cos\alpha+\operatorname{ch}(2nt\sin\alpha)}, \tag{466}$$

$$\psi=nt\cos\alpha+\tfrac{1}{2}\cos^{-1}\frac{1+\cos\alpha\operatorname{ch}(2nt\sin\alpha)}{\cos\alpha+\operatorname{ch}(2nt\sin\alpha)}$$

$$=nt\cos\alpha+{}^{1}\cos^{-1}\frac{2\cos\alpha}{1+z}. \tag{467}$$

At the same time, from (169),

$$\frac{d\varpi}{dt}=2n\frac{L}{M}\cdot\frac{z-z^2}{1-z^2}=2n\cos\alpha\frac{z}{1+z}$$

$$=n\frac{\cos 2\alpha+\cos\alpha\operatorname{ch}(2nt\sin\alpha)}{\cos\alpha+\operatorname{ch}(2nt\sin\alpha)}$$

$$=n\cos\alpha-\frac{n\sin^2\alpha}{\cos\alpha\operatorname{ch}(2nt\sin\alpha)}, \tag{468}$$

$$\varpi=nt\cos\alpha-\tfrac{1}{2}\cos^{-1}\frac{1+\cos\alpha\operatorname{ch}(2nt\sin\alpha)}{\cos\alpha+\operatorname{ch}(2nt\sin\alpha)}$$

$$=nt\cos\alpha-\tfrac{1}{2}\cos^{-1}\frac{2\cos\alpha}{1+z}; \tag{469}$$

so that, as in (167),

$$\psi-\varpi=\cos^{-1}\frac{2\cos\alpha}{1+z}=\sin^{-1}\frac{\sqrt{\{(1+z)^2-4\cos^2\alpha\}}}{1+z}. \tag{470}$$

Since $E=0$, $a=1$,

$$\frac{F^2\rho^2}{A^2n^2}=1-z^2, \tag{471}$$

so that the (α,β) or (ρ,ϖ) curve is given by

$$\varpi=nt\cos\alpha-\tfrac{1}{2}\cos^{-1}\frac{2\cos\alpha}{1+\sqrt{\left(1-\frac{F^2\rho^2}{A^2n^2}\right)}}$$

$$=nt\cos\alpha-\tfrac{1}{2}\cos^{-1}\frac{2\cos\alpha\left\{1-\sqrt{\left(1-\frac{F^2\rho^2}{A^2n^2}\right)}\right\}}{\frac{F^2\rho^2}{A^2n^2}}. \tag{472}$$

From equation (155),

$$F\frac{d\gamma}{dt}=\frac{F^2}{P}+An^2z^2$$

$$=\frac{F^2}{P}+An^2\left\{1-\frac{2\sin^2\alpha}{1+\cos\alpha\operatorname{ch}(2nt\sin\alpha)}\right\}^2, \tag{473}$$

or

$$\frac{d\gamma}{dt}=V-4\frac{An^2}{F}\frac{\sin^2\alpha}{1+\cos\alpha\operatorname{ch}(2nt\sin\alpha)}$$

$$+4\frac{An^2}{F}\frac{\sin^4\alpha}{\{1+\cos\alpha\operatorname{ch}(2nt\sin\alpha)\}^2}, \tag{474}$$

and integrating,

$$\gamma=Vt-\frac{An}{F}\frac{\sin 2\alpha\operatorname{sh}(2nt\sin\alpha)}{1+\cos\alpha\operatorname{ch}(2nt\sin\alpha)}, \tag{475}$$

thus determining α, β, γ as functions of the time t, an infinite time after the start.

If
$$L=0, \quad \cos\alpha=0, \quad \alpha=\tfrac{1}{2}\pi, \tag{476}$$

as if the body was fired from a smooth-bore gun without rotation; then

$$\frac{d\psi}{dt}=0, \quad \frac{d\varpi}{dt}=0,$$

while
$$\frac{dz}{dt}=n(1+z^2), \tag{477}$$

$$z=\tanh nt, \tag{478}$$

$$\rho=\frac{An}{F}\operatorname{sech} nt, \tag{479}$$

and the motion is confined to one plane.

45. When the body is projected sideways, in the direction OA, with velocity V, and with component angular velocities p and r about OA and OC, then initially,

$$u=V, \quad v=0, \quad w=0, \quad q=0, \tag{480}$$

so that
$$F=PV, \quad G=PVAp,$$

$$\frac{p}{n}=\frac{G}{AnF}=2\frac{L}{M}; \tag{481}$$

$$\begin{aligned}2T-Cr^2-\frac{F^2}{R}&=PV^2+Ap^2+Cr^2-\frac{P^2}{R}V^2-Cr^2\\&=-F^2\left(\frac{1}{R}-\frac{1}{P}\right)V^2+Ap^2\\&=-An^2a+4An^2\frac{L^2}{M^2};\end{aligned} \tag{482}$$

$$D=-a+4\frac{L^2}{M^2}, \quad E=-a+4\frac{B^2}{M^2}; \tag{483}$$

$$\begin{aligned}Z&=a(z^2-1)^2+a(z^2-1)-4\frac{L^2}{M^2}(z^2-1)-4\left(\frac{Bz-L}{M}\right)^2\\&=z\left\{az(z^2-1)-4\frac{B^2+L^2}{M^2}z+8\frac{BL}{M_2}\right\};\end{aligned} \tag{484}$$

so that this case comes under the head of those cases where a factor of Z is known; and here we have to put

$$\begin{aligned}\frac{\sigma-s}{M^2}&=\wp v-\wp u\\&=\tfrac{1}{4}a+\frac{B^2}{M^2}+\frac{BL}{M^2z}.\end{aligned} \tag{485}$$

When there is no rotation about OC, so that $r=0$, $B=0$,

$$Z=z^2\left\{a(z^2-1)-4\frac{L^2}{M^2}\right\}, \tag{486}$$

$$\frac{n\,dt}{dz}=\frac{1}{z\sqrt{\left\{a(z^2-1)-4\frac{L^2}{M^2}\right\}}}, \tag{487}$$

and therefore $z=0$ is the only solution for a prolate body, with $a=1$, since z^2-1 is negative; but

$$\frac{d^2z}{dt^2}\sim -n^2\left(1+4\frac{L^2}{M^2}\right)z, \tag{488}$$

so that the axis makes

$$\frac{n}{2\pi}\sqrt{\left(1+4\frac{L^2}{M^2}\right)} \tag{489}$$

small oscillations a second.

But with an oblate body, and $a=-1$,

$$n\frac{dt}{dz}=\frac{1}{z\sqrt{\left(1-4\frac{L^2}{M^2}-z^2\right)}}; \tag{490}$$

so that the form of the integral is different according as $1-4\frac{L^2}{M^2}$ is positive or negative.

When $1-4\frac{L^2}{M^2}$ is negative, the only real solution is again $z=0$, with a small nutation of the axis OC, making

$$\frac{n}{2\pi}\sqrt{\left(4\frac{L^2}{M^2}-1\right)} \tag{491}$$

oscillations a second.

But, with $1-4\frac{L^2}{M^2}$ positive,

$$nt=\frac{1}{\sqrt{\left(1-4\frac{L^2}{M^2}\right)}}\,\mathrm{ch}^{-1}\frac{\sqrt{\left(1-4\frac{L^2}{M^2}\right)}}{z}$$

$$z=\frac{\sqrt{\left(1-4\frac{L^2}{M^2}\right)}}{\mathrm{ch}\sqrt{\left(1-4\frac{L^2}{M^2}\right)}nt}. \tag{492}$$

Also

$$\frac{d\psi}{dz}=2\frac{L}{M}\frac{1}{z(1-z^2)\sqrt{\left(1-4\dfrac{L^2}{M^2}-z^2\right)}}$$

$$=2\frac{L}{M}\left(\frac{1}{z}+\frac{z}{1-z^2}\right)\frac{1}{\sqrt{\left(1-4\dfrac{L^2}{M^2}-z^2\right)}}$$

$$=2\frac{L}{M}n\frac{dt}{dz}+2\frac{L}{M}\frac{z}{(1-z^2)\sqrt{\left(1-4\dfrac{L^2}{M^2}-z^2\right)}}, \tag{493}$$

$$\psi=2\frac{L}{M}nt+\sin^{-1}\frac{2\dfrac{L}{M}}{\sqrt{(1-z^2)}}$$

$$=2\frac{L}{M}nt+\cos^{-1}\frac{\sqrt{\left(1-4\dfrac{L^2}{M^2}-z^2\right)}}{\sqrt{(1-z^2)}}. \tag{494}$$

With $B=0$, $E=1$, $a=-1$, and in (165),

$$\frac{F^2\rho^2}{A^2n^2}=z^2-1+E=z^2, \tag{495}$$

$$\rho=\frac{An}{F}z=\frac{An}{F}\sqrt{\left(1-4\frac{L^2}{M^2}\right)}\operatorname{sech}\sqrt{\left(1-4\frac{L^2}{M^2}\right)}nt, \tag{496}$$

and from (169),

$$\frac{d\varpi}{dz}=2\frac{L}{M}\frac{1}{\sqrt{Z}}=2\frac{L}{M}\frac{ndt}{dz}, \tag{497}$$

$$\varpi=2\frac{L}{M}nt, \tag{498}$$

so that

$$\rho=\frac{An}{F}\sqrt{\left(1-4\frac{L^2}{M^2}\right)}\operatorname{sech}\frac{\sqrt{\left(1-4\dfrac{L^2}{M^2}\right)}}{2\dfrac{L}{M}}\varpi, \tag{499}$$

a curve of the nature of a *separating* herpolhode.

At the same time, in (155),

$$F\frac{d\gamma}{dt}=\frac{F^2}{P}-An^2z^2$$

$$\frac{d\gamma}{dt}=V-\frac{An^2}{F}\left(1-4\frac{L^2}{M^2}\right)\operatorname{sech}^2\sqrt{\left(1-4\frac{L^2}{M^2}\right)}nt, \tag{500}$$

$$\gamma=Vt-\frac{An}{F}\sqrt{\left(1-4\frac{L^2}{M^2}\right)}\tanh\sqrt{\left(1-4\frac{L^2}{M^2}\right)}nt, \tag{501}$$

thus determining the motion of the body completely.

46. The discussion proceeds in the same manner when we put

$$p=0, \quad G=0, \quad L=0, \tag{502}$$

and now

$$Z=z^2\left\{a(z^2-1)-4\frac{B_2}{M^2}\right\}, \tag{503}$$

$$\frac{dnt}{dz}=\frac{1}{z\sqrt{\left\{a(z^2-1)-4\frac{B^2}{M^2}\right\}}}, \tag{504}$$

$$\frac{d\psi}{dz}=-2\frac{B}{M}\frac{z}{(1-z^2)\sqrt{Z}}$$

$$=-2\frac{B}{M}\frac{1}{(1-z^2)\sqrt{\left\{a(z^2-1)-4\frac{B^2}{M^2}\right\}}}; \tag{505}$$

and since

$$E=-a+4\frac{B^2}{M^2}, \tag{506}$$

$$\frac{d\varpi}{dz}=2\frac{B}{M}\frac{z}{(1+aE-z^2)\sqrt{Z}}$$

$$=2\frac{B}{M}\frac{1}{\left(4a\frac{B^2}{M^2}-z^2\right)\sqrt{\left\{a(z^2-1)-4\frac{B^2}{M^2}\right\}}}. \tag{507}$$

Considering only the case of the oblate body, $a=-1$, with $1-4\frac{B^2}{M^2}$ positive,

$$z=\sqrt{\left(1-4\frac{B^2}{M^2}\right)}\operatorname{sech}\sqrt{\left(1-4\frac{B^2}{M^2}\right)}nt, \tag{508}$$

$$\psi=\cos^{-1}\frac{2\frac{B}{M}z}{\sqrt{\left(1-4\frac{B^2}{M^2}\right)}\sqrt{(1-z^2)}}$$

$$=\sin^{-1}\frac{\sqrt{\left(1-4\frac{B^2}{M^2}-z^2\right)}}{\sqrt{\left(1-4\frac{B^2}{M^2}\right)}\sqrt{(1-z^2)}}, \tag{509}$$

$$\varpi=\cos^{-1}\frac{z}{\sqrt{\left(1-4\frac{B^2}{M^2}\right)}\sqrt{\left(4\frac{B^2}{M^2}+z^2\right)}}$$

$$=\sin^{-1}\frac{2\frac{B}{M}\sqrt{\left(1-4\frac{B^2}{M^2}-z^2\right)}}{\sqrt{\left(1-4\frac{B^2}{M^2}\right)}\sqrt{\left(4\frac{B^2}{M^2}+z^2\right)}}; \tag{510}$$

$$\frac{F^2\rho^2}{A^2n^2}=4\frac{B^2}{M^2}+z^2; \tag{511}$$

so that

$$\sqrt{\left(1-4\frac{B^2}{M^2}\right)}\frac{F\rho}{An}\sin\varpi=2\frac{B}{M}\sqrt{\left(1-\frac{F^2\rho^2}{A^2n^2}\right)},\tag{512}$$

or, putting $\frac{An}{F}=c$,

$$\frac{\alpha^2}{c^2}+\tfrac{1}{4}\frac{M^2}{B^2}\frac{\beta^2}{c^2}=1,\tag{513}$$

the equation of an ellipse.

The expression for V will be of the same nature as in (475).

This last state of motion can be illustrated experimentally by holding a disc of cardboard between the finger and thumb and flicking it sideways in the air.

47. The analysis developed here for the motion of Solid of a Revolution under No Forces in Infinite Frictionless Liquid is also available for other dynamical problems; for instance, in the exact treatment of the Precession and Nutation of the Earth's Axis, when, as in Poinsot's "Addition à la connaisance des temps," 1858, the mass of the disturbing Moon and Sun is supposed distributed in the form of a circular band or ring in the plane of the ecliptic, or else condensed into two equal repelling spheres, placed at the opposite poles of ecliptic at the same radial distance.

These equations are considered by Tisserand in the Comptes rendus, t. 101, 1885; as also Gylden's intermediate orbit, described under a central force varying partly as the distance and partly inversely as the square of the distance, which leads to similar analysis.

The motion of a ball rolling on a gravitating sphere, in which variations of internal density give rise to zonal harmonics of the first and second order in the expression of the potential, as well as the motion of a particle sliding on the smooth surface of a homogeneous gravitating ellipsoid of revolution, are further applications of the same analysis.

48. Taking the case of Precession and Nutation, in which the disturbance is due to a mass M, distributed in the form of a ring of radius R, the forces acting upon the Earth, an oblate spheroid of which the equatorial and polar moments of inertia are denoted by A and C, are equivalent to a couple round the line of nodes, of moment

$$\frac{3\gamma M}{R^3}(C-A)\sin\vartheta\cos\vartheta,\tag{514}$$

tending to decrease ϑ, where ϑ denotes the obliquity of the ecliptic, and γ the constant of gravitation; so that ψ denoting the longitude of the node, the equations of Energy and Momentum may be written

$$\tfrac{1}{2}A\frac{d\vartheta^2}{dt^2}+\tfrac{1}{2}A\sin^2\vartheta\frac{d\psi^2}{dt^2}=-\tfrac{3}{2}\frac{\gamma M}{R^3}(C-A)\sin^2\vartheta+H, \tag{515}$$

$$A\sin^2\vartheta\frac{d\psi}{dt}+Cr\cos\vartheta=G, \tag{516}$$

where H and K are the constants of the problem, and r the constant angular velocity of the Earth about its axis.

Putting $\cos\vartheta=z$ and eliminating $\frac{d\psi}{dt}$ between (515) and (516),

$$\begin{aligned}\frac{dz^2}{dt^2}&=-3\frac{\gamma M}{R^3}\frac{C-A}{A}(1-z^2)^2+\frac{2H}{A}(1-z^2)-\left(\frac{G-Crz}{A}\right)^2\\&=n^2\left\{-(z^2-1)(z^2-1+D)-\left(\frac{Crz-G}{An}\right)^2\right\}\\&=n^2Z,\end{aligned} \tag{517}$$

on putting

$$\frac{3\gamma M}{R^3}\frac{C-A}{A}=n^2,\quad \frac{2H}{A}=n^2D, \tag{518}$$

and then

$$\frac{d\psi}{dt}=\frac{G-Crz}{A\sin^2\vartheta}, \tag{519}$$

$$\frac{d\psi}{dz}=\frac{G-Crz}{An(1-z^2)\sqrt{Z}}, \tag{520}$$

equations of exactly the same form as those required for the motion of an oblate solid in liquid, so that the previous pseudo-elliptic solutions are immediately available; for instance, the algebraical solutions given by equations (345) and (358).

To gain some idea of the actual magnitude of the constants in this problem in the actual case of the Earth, let μ denote the mean angular velocity of Precession and ω the mean obliquity of the ecliptic, then (Quarterly Journal of Mathematics, vol. XIV, p. 173)

$$\mu=\frac{3\gamma M}{2R^3}\frac{C-A}{Cr}\cos\omega, \tag{521}$$

so that

$$n^2=2\frac{C}{A}r\mu\sec\omega; \tag{522}$$

and in this equation we may take

$$\frac{C}{A}=1, \quad \omega=23^\circ\ 27', \tag{523}$$

and, with a year as unit of time, and an annual Precession of $50''.25$,

$$r=2\pi\times 366, \quad \mu=\frac{50.25}{206265}; \tag{524}$$

this makes

$$n=1.115 \text{ and } \frac{Cr}{An}=2062. \tag{525}$$

But if we try to utilize the algebraical case given by equation (345) we find from (343),

$$\frac{Cr}{An}=2\frac{B}{M}=\frac{H}{K}=\frac{H}{\sqrt{(H^2+1)}}, \tag{526}$$

which, with $H>1$, must be <0.707, so that the angular velocity of the Earth would require to be reduced to about one three-thousandth of its present amount for this motion to be possible; and now the path of the pole in the sky would give rise to interesting astronomical speculations.

49. In Gylden's orbit, with polar coordinates $\rho=\frac{1}{u}$ and ϖ, the central force

$$P=\mu\rho+\frac{\mu'}{\rho^3}; \tag{527}$$

or, more generally,

$$P=a\rho+2b-\frac{4d}{\rho^3}, \tag{528}$$

suppose; so that

$$\tfrac{1}{2}\frac{d\rho^2}{dt^2}+\tfrac{1}{2}\rho^2\frac{d\varpi^2}{dt^2}=\tfrac{1}{2}a\rho^2+2b\rho+3c+\frac{4d}{\rho}, \tag{529}$$

and

$$\rho^2\frac{d\varpi}{dt}=h. \tag{530}$$

Thence

$$\begin{aligned}\rho^2\frac{d\rho^2}{dt^2}&=a\rho^4+4b\rho^3+6c\rho^2+4d\rho-h^2,\\&=R,\end{aligned} \tag{531}$$

suppose; so that

$$\frac{d\varpi}{d\rho}=\frac{h}{\rho\sqrt{R}}, \text{ or } \frac{d\varpi}{du}=\frac{-hu}{\sqrt{(-h^2u^4+4du^3+6cu^2+4bu+a)}} \tag{531a}$$

is the differential relation for the orbit, and thus pseudo-elliptic cases can be constructed, immediately from Abel's analysis (*Œuvres complètes*).

50. For the motion of a ball, of radius b, rolling on a sphere of radius $R-b$, under a zonal potential denoted by V, the equations of Energy and Momentum become

$$\tfrac{1}{2}\left(1+\frac{k^2}{b^2}\right)R^2\left(\frac{d\vartheta^2}{dt^2}+\sin^2\vartheta\,\frac{d\psi^2}{dt^2}\right)=V+H, \tag{532}$$

$$\left(1+\frac{k^2}{b^2}\right)\sin^2\vartheta\,\frac{d\psi}{dt}+\frac{k^2}{bR}\,r\cos\vartheta=G, \tag{533}$$

where k denotes the radius of gyration of the ball.

The variable part of V depending on $\cos\vartheta$ and $\cos^2\vartheta$ or $\sin^2\vartheta$, we can bring our equations into the same shape as before by putting

$$V+H=\tfrac{1}{2}n^2R^2\left(1+\frac{k^2}{b^2}\right)(a\sin^2\vartheta+4b'\cos\vartheta+D), \tag{534}$$

and

$$G=2n\left(1+\frac{k^2}{b^2}\right)\frac{L}{M},\quad \frac{k^2r}{bR}=2n\left(1+\frac{k^2}{b^2}\right)\frac{B}{M}, \tag{535}$$

so that

$$\frac{d\vartheta^2}{dt^2}+\sin^2\vartheta\,\frac{d\psi^2}{dt^2}=n^2(a\sin^2\vartheta+4b'\cos\vartheta+D), \tag{536}$$

$$\sin^2\vartheta\,\frac{d\psi}{dt}=2n\,\frac{L-B\cos\vartheta}{M}. \tag{537}$$

Eliminating $\frac{d\psi}{dt}$ and putting $\cos\vartheta=z$, we obtain as before

$$\frac{dz}{dt}=n\sqrt{Z},$$

where

$$Z=a(1-z^2)^2+(4b'z+D)(1-z^2)-4\left(\frac{L-Bz}{M}\right)^2 \tag{538}$$

and

$$\frac{d\psi}{dz}=2\,\frac{L-Bz}{M(1-z^2)\sqrt{Z}}, \tag{539}$$

of which the special case for $b'=0$ has received consideration, and the solution proceeds as formerly.

When the ball is projected without any spin, so that $r=0$, the equations of motion are the same as for a particle on a smooth sphere, in the same field of force; and now, with $b'=0$,

$$Z=a(1-z^2)(1-z^2+aD)-4\frac{L^2}{M^2}. \tag{540}$$

With a positive and equal to $+1$, we can put

$$\frac{dz^2}{dt^2}=n^2(\alpha^2-z^2)(\beta^2-z^2), \tag{541}$$

and taking

$$\beta>1>\alpha>z>-\alpha>-1>\beta, \tag{542}$$

the particle or ball crosses the equator, and

$$nt=\int\frac{dz}{\sqrt{\{(\alpha^2-z^2)(\beta^2-z^2)\}}}$$

$$=\frac{1}{\beta}\,\mathrm{sn}^{-1}\left(\frac{z}{\alpha},\ \frac{\alpha}{\beta}\right), \tag{543}$$

or

$$z=\alpha\ \mathrm{sn}\ \beta nt. \tag{544}$$

Putting $z^2=1$ makes

$$4\frac{L^2}{M^2}=(\beta^2-1)(1-\alpha^2), \tag{545}$$

so that

$$\frac{d\psi}{dz}=\frac{\sqrt{\{(\beta^2-1)(1-\alpha^2)\}}}{(1-z^2)\sqrt{\{(\alpha^2-z^2)(\beta^2-z^2)\}}}. \tag{546}$$

Now if we put

$$\frac{z^2}{\alpha^2}=x \text{ and } \frac{1}{\alpha^2}=a,\quad k^2=\frac{\alpha^2}{\beta^2}; \tag{547}$$

$$\psi=\tfrac{1}{2}\int\frac{\sqrt{\{-a(1-a)(1-k^2a)\}}}{a-x}\ \frac{dx}{\sqrt{\{x(1-x)(1-k^2x)\}}}$$

$$=\tfrac{1}{2}\int\frac{\sqrt{(-A)}\,dx}{(a-x)\sqrt{X}}, \tag{548}$$

where

$$X=x(1-x)(1-k^2x), \tag{549}$$

and A is the value of X when a replaces x; this is a canonical form of the Elliptic Integral of the Third Kind, with the appropriate multiplier $\sqrt{(-A)}$ in the numerator.

With a negative and equal to -1, we can have

$$Z=(\alpha^2-z^2)(z^2\mp\beta^2). \tag{550}$$

With

$$Z=(\alpha^2-z^2)(z^2-\beta^2), \tag{551}$$

the path of the particle is bounded by two parallels of latitude on the same side of the equator, and

$$1>\alpha>z>\beta>0>-\beta>z>-\alpha>1; \tag{552}$$

$$nt = \int \frac{dz}{\sqrt{\{(\alpha^2 - z^2)(z^2 - \beta)^2\}}}$$

$$= \frac{1}{\alpha}\,\mathrm{dn}^{-1}\left\{\frac{z}{\alpha},\ \sqrt{\left(1 - \frac{\beta^2}{\alpha^2}\right)}\right\}, \tag{553}$$

$$z = \alpha\ \mathrm{dn}\ \alpha nt, \tag{554}$$

and

$$4\frac{L^2}{M^2} = (1 - \alpha^2)(1 - \beta^2), \tag{555}$$

$$\frac{d\psi}{dz} = \frac{\sqrt{\{(1 - \alpha^2)(1 - \beta^2)\}}}{(1 - z^2)\sqrt{\{(\alpha^2 - z^2)(z^2 - \beta^2)\}}}. \tag{556}$$

Putting

$$\frac{z^2}{\alpha^2} = x,\quad \frac{1}{\alpha^2} = a,\quad 1 - \frac{\beta^2}{\alpha^2} = k^2, \tag{557}$$

makes

$$\psi = \tfrac{1}{2}\int \frac{\sqrt{(-A)}\,dx}{(a - x)\sqrt{X}}, \tag{558}$$

where

$$X = x(1 - x)(x - k'^2). \tag{559}$$

With

$$Z = (\alpha^2 - z^2)(z^2 + \beta^2), \tag{560}$$

the particle again crosses the equator, and

$$nt = \int \frac{dz}{\sqrt{\{(\alpha^2 - z^2)(z^2 + \beta^2)\}}}$$

$$= \frac{1}{\sqrt{(\alpha^2 + \beta^2)}}\,\mathrm{cn}^{-1}\left\{\frac{z}{\alpha},\ \frac{\alpha}{\sqrt{(\alpha^2 + \beta^2)}}\right\}, \tag{561}$$

$$z = \alpha\ \mathrm{cn}\ \sqrt{(\alpha^2 + \beta^2)}\,nt, \tag{562}$$

$$4\frac{L^2}{M^2} = (1 - \alpha^2)(1 + \beta^2), \tag{563}$$

and with

$$\frac{z^2}{\alpha^2} = x,\quad \frac{1}{\alpha^2} = a,\quad \frac{\alpha^2}{\alpha^2 + \beta^2} = k^2, \tag{564}$$

$$\psi = \tfrac{1}{2}\int \frac{\sqrt{(-A)}\,dx}{(a - x)\sqrt{X}}, \tag{565}$$

where

$$X = x(1 - x)\left(x + \frac{k'^2}{k^2}\right). \tag{566}$$

Thence pseudo-elliptic cases can easily be constructed, but the secular term cannot be cancelled.

But when the ball is projected with appropriate spin r, we have an additional constant at our disposal, and it is possible to make the path an algebraical curve; thus, for instance, from (345), the equations

$$\sin^3\theta\cos 3\psi = H\cos^3\theta - 1, \tag{567}$$

$$\sin^3\theta\sin 3\psi = \cos\theta\sqrt{\{-(H^2+1)\cos^4\theta + 3\cos^2\theta + 2H\cos\theta - 3\}}, \tag{568}$$

represent a possible algebraical trajectory of the centre of the ball, rolling on this gravitating sphere, the circumstances of the initial projection being chosen appropriately. So also equations (358).

51. In the motion of a particle on a smooth, homogeneous, gravitating ellipsoid, bounded by the surface of revolution about $O\gamma$ of an ellipse

$$\frac{\rho^2}{\alpha^2} + \frac{\gamma^2}{\beta^2} = 1, \tag{569}$$

referred to $O\rho$ and $O\gamma$ as coordinate axes, the variable part of the potential on the surface may be put equal to $\frac{1}{2}A\rho^2$, so that, with cylindrical coordinates ρ, ϖ, γ, the equations of Momentum and Energy may be written

$$\rho^2\frac{d\varpi}{dt} = K, \tag{570}$$

$$\frac{d\rho^2}{dt^2} + \rho^2\frac{d\varpi^2}{dt^2} + \frac{d\gamma^2}{dt^2} = A\rho^2 + B. \tag{571}$$

From (569) we find

$$\frac{d\gamma}{dt} = \frac{\beta}{\alpha}\,\frac{\rho}{\sqrt{(\alpha^2-\rho^2)}}\,\frac{d\rho}{dt}, \tag{572}$$

so that

$$\frac{d\rho^2}{dt^2} + \frac{K^2}{\rho^2} + \frac{\beta^2}{\alpha^2}\,\frac{\rho^2}{\alpha^2-\rho^2}\,\frac{d\rho^2}{dt^2} = A\rho^2 + B,$$

or

$$\left\{\alpha^2 - \left(1-\frac{\beta^2}{\alpha^2}\right)\rho^2\right\}\rho^2\frac{d\rho^2}{dt^2} = (\alpha^2-\rho^2)(A\rho^4 + B\rho^2 - K^2). \tag{573}$$

Putting $\rho^2 = \alpha^2 z$,

$$\frac{dz^2}{dt^2} = 4\,\frac{(1-z)\left(Az^2 + \dfrac{B}{\alpha^2}z - \dfrac{K^2}{\alpha^4}\right)}{1-\left(1-\dfrac{\beta^2}{\alpha^2}\right)z}$$

$$= 4\,\frac{Z}{\left\{1-\left(1-\dfrac{\beta^2}{\alpha^2}\right)z\right\}^2}, \tag{574}$$

where

$$Z=(z-1)\left\{\left(1-\frac{\beta^2}{\alpha^2}\right)z-1\right\}\left(Az^2+\frac{B}{\alpha^2}z-\frac{K^2}{\alpha^4}\right) \tag{575}$$

and

$$z\frac{d\varpi}{dt}=\frac{K}{\alpha^2}, \tag{576}$$

so that

$$\frac{d\varpi}{dz}=\tfrac{1}{2}\frac{K}{\alpha^2}\,\frac{1-\left(1-\dfrac{\beta^2}{\alpha^2}\right)z}{z\sqrt{Z}}, \tag{577}$$

and this can be treated in the same manner as the previous equations (531) for Gylden's orbit.

ARTILLERY COLLEGE, WOOLWICH, 27 *April*, 1897.

Surfaces of Rotation with Constant Measure of Curvature and their Representation on the Hyperbolic (Cayley's) Plane.

BY GEO. F. METZLER, *Odessa, Ont.*

In Crelle, vols. 19 and 20, Minding has shown that it is easy to obtain the formulas which express the relations between the sides and angles of a triangle of which the sides are geodesic lines on a surface of rotation with constant measure of curvature.

It is only necessary to replace the radius a of the sphere in the formulas of the ordinary spherical trigonometry by $a\sqrt{-1}$.

That this is also true for the formula expressing the area of the triangle is not stated, and as that is the formula which concerns us here, a proof of the same will be given by means of polar and rectangular coordinates. I do not know that such proof has yet been published.

It will be assumed that such definitions and fundamental equations as are to be found in Salmon's "Geometry of Three Dimensions," Joachimsthal's "Anwendung der Differentiale Rechnung" and similar works, are well known.

If the equations of the surface are

$$u = f(r) - z = 0, \quad r^2 = x^2 + y^2, \tag{1}$$

and we designate the derivates of u as follows:

$$p = \frac{\partial u}{\partial x}, \quad q = \frac{\partial u}{\partial y}, \quad -1 = \frac{\partial u}{\partial z}, \quad v = \frac{\partial^2 u}{\partial x^2},$$
$$s = \frac{\partial^2 u}{\partial x \partial y}, \quad t = \frac{\partial^2 u}{\partial y^2} \text{ and } f' = \frac{\partial z}{\partial r} = \frac{\partial u}{\partial r},$$

then R_1 and R_2, the principal radii of curvature, are the roots of the equation

$$R^2(vt - s^2) - R\,[(1+q^2)\,v - 2pqs + (1+p^2)\,t]\,\sqrt{1+p^2+q^2} + (1+p^2+q^2)^2 = 0. \quad (2)$$

Thus the measure of curvature $\dfrac{1}{R_1 R_2}$ is expressed by

$$\frac{1}{R_1 R_2} = \frac{vt - s^2}{(1+p^2+q^2)^2}. \quad (3)$$

From (1) we obtain

$$p = \frac{x}{r} f', \quad q = \frac{y}{r} f', \quad v = \frac{x^2}{r^2} f'' + \frac{y^2}{r^3} f',$$

$$s = \frac{xy}{r^2} f'' - \frac{xy}{r^3} f', \quad t = \frac{y^2}{r^2} f'' + \frac{x^2}{r^3} f'.$$

Then

$$vt - s^2 = \left(\frac{x^2}{r^2} f'' + \frac{y^2}{r^3} f'\right)\left(\frac{y^2}{r^2} f'' + \frac{x^2}{r^3} f'\right) - \left(\frac{xy}{r^2} f'' - \frac{xy}{r^3} f'\right)^2 = \frac{f' f''}{r},$$

$$(1+p^2+q^2)^2 = (1+f'^2)^2.$$

Then follows

$$\frac{1}{R_1 R_2} = \frac{f' f''}{r\,(1+f'^2)^2} = -\frac{1}{2r}\,\frac{d}{dr}\left(\frac{1}{1+f'^2}\right). \quad (4)$$

If the surface has a constant measure of curvature, $\pm \dfrac{1}{a^2}$ say, then it follows from (4) that

$$-\frac{1}{1+f'^2} = \pm \int \frac{2r\,dr}{a^2} = \pm \frac{r^2}{a^2} - b^2,$$

where $-b^2$ is the constant of integration. Then

$$1 + f'^2 = \frac{a^2}{a^2 b^2 \mp r^2},$$

$$f'^2 = \frac{a^2(1-b^2) \pm r^2}{a^2 b^2 \mp r^2},$$

and

$$dz = dr\sqrt{\frac{a^2(1-b^2) \pm r^2}{a^2 b^2 \mp r^2}}. \quad (5)$$

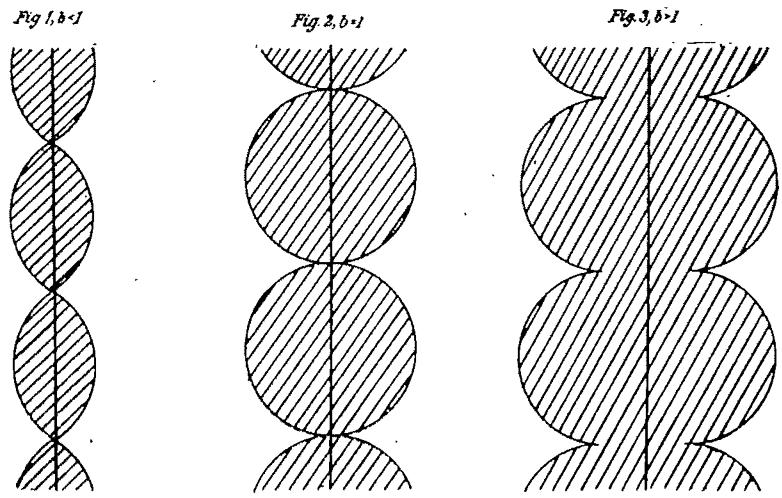
Equation (5) gives the meridian curves of six different forms of surfaces of rotation. According to the values given to b^2 we obtain one of three different forms

by using the upper signs, and also one of three other different forms when the lower signs are used.

Conversely, it is easy to show that all surfaces of rotation whose meridian curves are given by (5) are surfaces of constant curvature.

Sections of the surfaces through the axis of rotation and a pair of meridian curves would appear as in the following figures.*

Surfaces with Positive Measure of Curvature.



From the equation

$$\frac{dz}{dr} = \sqrt{\frac{a^2(1-b^2)+r^2}{a^2b^2-r^2}}$$

we see that the greatest real value for r^2 is a^2b^2, for which $dr = 0$. dz vanishes for $r^2 = a^2b^2 - a^2$. In Fig. 1, however, this value of r is imaginary, and 0 is the minimum real value of r.

Thus the surfaces consist of an infinite number of parts, each part having the axis of z as axis of rotation. The area of one part of these surfaces is given by

$$\left.\begin{aligned} &2\int_{r^2=a^2b^2-a^2}^{r^2=a^2b^2} 2\pi r ds, \quad ds = \sqrt{1+\left(\frac{dz}{dr}\right)^2}\,dr, \\ &2\int 2\pi r ds = 4\pi\int_{a^2b^2-a^2}^{a^2b^2} \frac{ar dr}{\sqrt{a^2b^2-r^2}} = -\left[4\pi a\sqrt{a^2b^2-r^2}\right]_{a^2b^2-a^2}^{a^2b^2} \\ &\qquad = 4\pi a^2. \end{aligned}\right\} \qquad (6)$$

* Klein, Nicht-Euklidische Geometrie, p. 186.

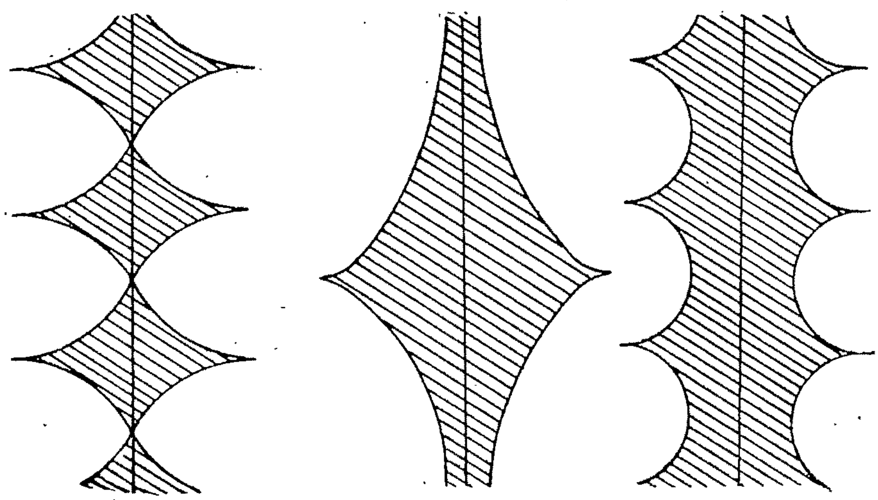
If in Fig. 1 we do not consider the imaginary part of the surface, 0 becomes the inferior limit and the area is

$$4\pi a^2 b,$$

and this is equal to the superficial area of a cylinder of altitude $2a$ circumscribing the surface. It is easy to see that the meridian curves approach in form for Fig. 1 those on a prolate spheroid, for Fig. 3 those on an oblate spheroid, while for Fig. 2 they are circular.

Surfaces with Constant Negative Measure of Curvature.

Fig. 4, $b^2 > 0$ Fig. 5, $b^2 = 0$ Fig. 6, $b^2 < 0$



The equation for these forms is

$$\frac{dz}{dr} = \sqrt{\frac{a^2(1-b^2)-r^2}{a^2b^2+r^2}},$$

which is the same as that for the other forms with $-a^2$ written for a^2.

In Figs. 4 and 6 we have also a series of similar parts, while in Fig. 5 there is only one part extending to infinity in both directions. It is easy to show that the area of the surface of each part is the same for all cases when we take for limits values of r for which dz and dr vanish. It is

$$4\pi a^2.$$

Otherwise, if we take in Fig. 4 zero as the inferior limit of integration, the area is

$$4\pi a^2(1-b).$$

Thus the area of the real part of Fig. 4 equals the area of the imaginary part of Fig. 1, and *vice versa.*

Now let us seek the formula for the area of any triangle whose sides are geodesics on these surfaces.

When we consider the geodesics projected on a plane perpendicular to the axis of rotation, the equation of the projection in polar coordinates r, ϕ is (compare Joachimsthal, "Anwendung der Diff. Rech.," p. 172)

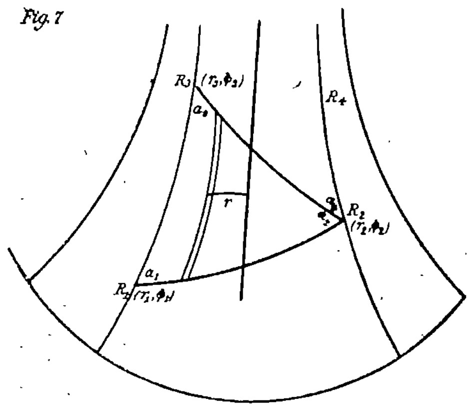
$$\phi = \nu \int \frac{dr}{r} \sqrt{\frac{1+f'^2}{r^2-\nu^2}}, \tag{7}$$

where
$$1+f'^2 = 1+\left(\frac{dz}{dr}\right)^2 = \frac{a^2}{a^2b^2 \mp r^2}.$$

Then
$$d\phi = \frac{a\nu\, dr}{r} \sqrt{\frac{1}{(a^2b^2 \mp r^2)(r^2-\nu^2)}}. \tag{8}$$

Through each angle of any triangle a meridian passes. If no side of the triangle be a meridian, the triangle may be divided into two, of which one side is the dividing meridian. Thus we need consider only triangles one of whose sides is a meridian. Let $R_1 R_2 R_3$ be such a triangle.

Fig. 7



Let also the coordinates of R_1, R_2 and R_3 be designated by $(r_1 \phi_1)$, $(r_2 \phi_2)$,

and $(r_3 \phi_3)$ and r_{12} designate the values of r along the side $R_1 R_2$, r_{23} its values along $R_2 R_3$. The area Δ of the triangle $R_1 R_2 R_3$ is given by

$$\Delta = \int\int r d\phi \, ds = \int_{\phi_1}^{\phi_3} \int_{r_{23}}^{r_{12}} r d\phi \frac{adr}{\sqrt{a^2 b^2 \mp r^2}}$$
$$= \mp a \int_{\phi_1}^{\phi_3} \sqrt{a^2 b^2 \mp r_{12}^2} \, d\phi \pm a \int_{\phi_1}^{\phi_3} \sqrt{a^2 b^2 \mp r_{23}^2} \, d\phi .$$

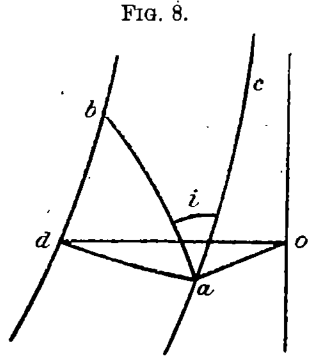
Substituting for $d\phi$ from (8),

$$\Delta = \mp a^2 \int_{r_1}^{r_2} \frac{\nu_{12} \, dr_{12}}{r_{12} \sqrt{r_{12}^2 - \nu_{12}^2}} \pm a^2 \int_{r_2}^{r_3} - \frac{d\left(\frac{\nu_{23}}{r_{23}}\right)}{\sqrt{1 - \frac{\nu_{23}^2}{r_{23}^2}}}$$
$$= \pm a^2 \left[\sin^{-1} \frac{\nu_{12}}{r_2} - \sin^{-1} \frac{\nu_{12}}{r_1} - \sin^{-1} \frac{\nu_{23}}{r_2} + \sin^{-1} \frac{\nu_{23}}{r_3} \right]. \tag{9}$$

If we take on such a surface two meridian curves ac and db infinitely near together,

$$od = oa = r,$$
$$ab = ds,$$

FIG. 8.



angle $\quad aod = d\phi, \quad ad = rd\phi = ab \cos bad$

$$= ab \sin cab$$
$$= ds \sin i,$$
$$\nu ds = r^2 d\phi \text{ (see Joachimsthal } \ldots\text{.)}.$$

where ν is so chosen that it is positive.

When
$$ds = r d\phi, \quad \nu = r,$$
then $i = 90°$. Thus ν is the minimum value of r, i. e. its value where the geodesic crosses the meridian perpendicularly.

$$\nu ds = rab \sin i = rds \sin i.$$

Therefore
$$\sin^{-1} \frac{\nu}{r} = i. \tag{10}$$

Introducing this into (3), we have the interesting result, α_1, α_2, α_3, being the angles of the triangle,

$$\begin{aligned} \Delta &= \pm a^2 \left[\alpha_2 + \alpha_4 - (\pi - \alpha_1) - \alpha_4 + \alpha_3\right], \\ \Delta &= \pm a^2 \left[\alpha_1 + \alpha_2 + \alpha_3 - \pi\right]. \end{aligned} \tag{11}$$

Thus for all these surfaces of positive measure of curvature the area of a triangle is given by the same formula as holds for the sphere, and when the curvature is negative the proper formula is obtained by replacing a^2 by $-a^2$. (This result is the same as has been found by Beltrami by means of curvilinear coordinates, Beltrami's "Saggio.")

Before considering the conformal representation of these lines, areas, and surfaces, let us study more closely the situation of the geodesics on the surfaces. The case in which the meridian curve is the tractrix (Fig. 5) is simplest and also the most interestng.

Equation (8), the equation of projection, becomes for the traction

$$d\phi = \frac{a\nu\, dr}{r^2\sqrt{r^2 - \nu^2}},$$

then
$$\phi = \frac{1}{\nu}\sqrt{a^2 - \nu^2} - \frac{a}{r\nu}\sqrt{r^2 - \nu^2} + C.$$

Let λ be the angle that the projection of the geodesic makes with the radius r. Then for any of the surfaces with negative measure of curvature,

$$\operatorname{cotan} \lambda = \frac{dr}{r d\phi} = \frac{\sqrt{(a^2b^2 + r^2)(r^2 - \nu^2)}}{a\nu}.$$

When $r^2 = a^2(1 - b^2)$ we have an edge of the surface (cuspidal edge). Let λ_0 be the value of λ at this edge, then

$$\sin \lambda_0 = \frac{\nu}{a\sqrt{1 - b^2}}. \tag{12}$$

From every point of this edge geodesics proceed in all directions, each having a definite value of λ_c and consequently a definite value for ν. From the equation for cotan λ we see that ν is the minimum value of r, for which value $\lambda = \frac{\pi}{2}$.

For every value of r from ν to $a\sqrt{1-b^2}$ we obtain a definite ϕ, and therefore a definite x and y, and then from the equation of the surface the corresponding z, so that it is not difficult to form the geodesics on the surface.

Let us now consider the schaar of geodesics which reach their highest points on a certain meridian OA. From this meridian the geodesics considered proceed symmetrically in both directions. Let $\phi = 0$ for this meridian. On it $r = \nu$, $\lambda = \frac{\pi}{2}$ and dr must be positive for each curve. For the surface (Fig. 5) $b^2 = 0$ and

$$\phi = \pm \frac{a}{\nu r}\sqrt{r^2 - \nu^2} = \frac{a^2}{r^2} \text{ cotan } \lambda. \tag{13}$$

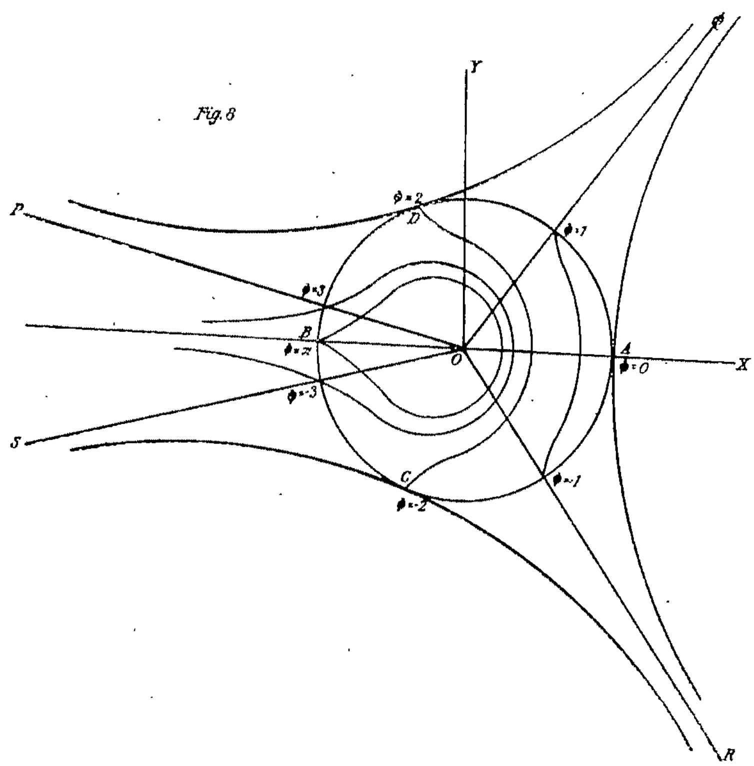


Fig. 8

When ν is very small, λ_c is very small, and for finite r, $d\phi$ is very small, and the geodesic keeps near the meridian; but as r approaches zero, $d\phi$

increases rapidly, so that the curve winds infinitely often around the surface before reaching its highest point.

Let us suppose the upper half of the surface projected on the xy plane passing through the cuspidal edge. This edge projects into the circle $ACBD$, the axis of rotation into the point O, centre of the circle, and the meridian OA into the radius $OA = a$.

From (13) we see that r is infinite when $\pm\phi = \frac{a}{\nu}$. When $\nu = a$ the geodesic touches the cuspidal edge at A and proceeds to infinity, having the lines for which $\phi = \pm 1$ as asymptotes. This is a point to be specially noted in order that correct figures be drawn. It may be stated thus:

Two geodesics which touch the cuspidal edge and meet at infinity have points of contact subtending an angle $\frac{360°}{\pi}$ or 2 at the centre O. Thus the curves touching where $\phi = 0$ and $\phi = 2$ would have the line for which $\phi = 1$ as asymptote. Let $r = a$, then $\nu = a \sin \lambda_c$ and $\phi = \operatorname{cotan} \lambda_c$, then for

$$\left.\begin{array}{llll} \nu = a & , \ \phi = 0 & , \ \lambda_c = 90° & , \\ \nu = \frac{a}{\nu^2} & , \ \phi = 1 & , \ \lambda_c = 45° & , \\ \nu = \frac{a}{2} & , \ \phi = \sqrt{3}, & \lambda_c = 30° & , \\ \nu = a \sin 17° \tfrac{2}{3}, & \phi = \pi & , \ \lambda_c = 17° \tfrac{2}{3}. & \text{(This last is approximate.)} \end{array}\right\} \quad (15)$$

The condition for a point of inflexion is (compare Williamson's "Diff. Cal.," p. 278)

$$rf'' - 2f'^2 - r^2 = 0,$$

when
$$r = \frac{a\nu}{(a^2 - \nu^2\phi^2)^{\frac{1}{2}}} = f(\phi), \quad f' = \frac{a\nu^3\phi}{(a^2 - \nu^2\phi^2)^{\frac{3}{2}}},$$
$$f'' = \frac{a^3\nu^3 + 2a\nu^5\phi^2}{(a^2 - \nu^2\phi^2)^{\frac{5}{2}}}.$$

Therefore
$$a^2\nu^2 = (a^2 - \nu^2\phi^2)^2,$$
$$= \frac{a^4\nu^4}{r^4}$$

and
$$r^2 = a\nu = a^2 \sin\lambda.$$

Thus these geodesics have in general a point of inflection for $r = \sqrt{a\nu} = a\sqrt{\sin\lambda}$.

They are touched at this point by the curves whose tangents are the inflexional tangents of the surface. Let r_t, ϕ be the coordinates of the projection of these curves, while r_g, ϕ are the coordinates of projection of the geodesics. From sections 377, 380 and 381, Salmon's "Geometry of Three Dimensions," the equation of these curves, viz.

$$d\phi = \frac{adr_t}{r_t\sqrt{a^2 - r_t^2}}$$

can be derived.

For the same values of r, ϕ we have for the geodesics

$$d\phi = \frac{\nu adr_g}{r_g^2\sqrt{r_g^2 - \nu^2}}.$$

From these equations it follows that

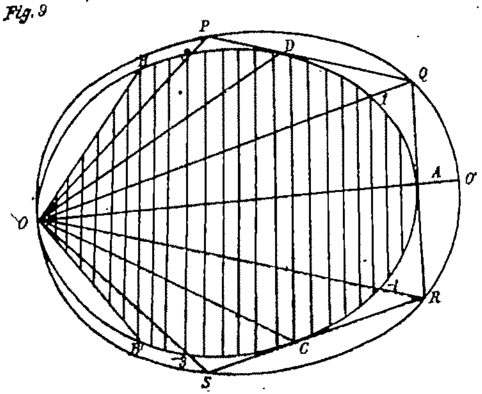
$$\frac{dr_g}{dr_t} = \frac{r\sqrt{r^2 - \nu^2}}{\nu\sqrt{a^2 - r^2}}.$$

This fraction takes all values from 0 to ∞ as r increases from ν to a, and equals unity when $r^2 = a\nu$, that is, the two curves touch for this value of r. (This result has been obtained by Mr. Bacharack from another point of view.)

Let us consider the conformal representation of the upper half of this surface on Cayley's plane of hyperbolic measurement.

Let the surface be cut along the meridian which is projected in OB (Fig. 8). The two edges thus formed will be represented by OB and OB'. Since the meridian curves all meet the axis of rotation in the point O for which $z = \infty$, its

Fig. 9



representative O in Fig. 9 should lie on the boundary. (Compare "Nicht-

Euklidische Geometrie," Klein. In fact these celebrated lectures form the source of my knowledge on these questions prior to this investigation.)

Let $OPQRS$ be the boundary of the plane, and $BDACB'$ represent the cuspidal edge. Angles remain unchanged, and geodesics, including meridians, are represented by straight lines. P, Q, R and S represent points infinitely distant where, in Fig. 8, tangent curves at D, A and C touch OP, OQ, OR and OS. Thus the hatched part of the figure is the representation of the upper half of the surface of rotation of the tractrix.

As the area is $2\pi a^2$, when a is small the representation should occupy but a small part of the plane. By moving all the lettered points proportionally towards O', the figure can be made small, care being taken that the curve $BDACB'$ touches the lines PQ, QR, RS at D, A and C respectively. With this figure there is no difficulty in calculating areas.

The formula for a triangle with angles α, β, γ is ("Nicht-Euklidische Geometrie," p. 121)

$$\text{Area} = 4x^2(\pi - \alpha - \beta - \gamma).$$

From Fig. 8 or equations (15) we see that each geodesic subtending an angle 2 at the centre O makes angles equal to $\frac{\pi}{4}$ with radii or meridians through the points where it cuts the cuspidal edge. Then the angles of the triangle OAD are respectively $O, \frac{\pi}{4}, \frac{\pi}{4}$ and the area of this triangle is $4x^2\left(\pi - O - \frac{\pi}{2}\right)$ or

$$2\pi x^2.$$

The area of the corresponding triangle on the surface is $\frac{\pi}{2}a^2$, and therefore corresponding areas are to each other as $4x^2$ to a^2.

The whole area is

$$2\pi(4x^2) = 8\pi x^2.$$

The area of the sector $OA1D$ is $8x^2$, which is greater than the triangle OAD, and the area of $OAQD$ is

$$4x^2\left(2\pi - 2\frac{\pi}{2}\right) = 4x^2\pi,$$

and this is greater than the sector, and so for other parts of the figure consistent results are obtained. Such results are impossible when a figure such as $OA1D$ is taken to represent the surface. It is in fact $\frac{1}{\pi}$th of the surface.

*Sur les Méthodes d'Approximations Successives dans la Théorie des Équations Différentielles.**

PAR ÉMILE PICARD.

J'ai consacré plusieurs Mémoires à l'application de méthodes d'approximations successives pour démontrer l'existence et faire la recherche des intégrales de certaines équations différentielles, quand des conditions aux limites sont données qui définissent ces intégrales. Ces méthodes s'appliquent aux équations différentielles ordinaires comme aux équations aux dérivées partielles, mais pour ces dernières les conditions d'application sont bien différentes suivant que les équations considérées appartiennent au type elliptique ou au type hyperbolique. Les premières se rencontrent surtout en Physique mathématique et dans la théorie des fonctions ; je ne m'en occuperai pas dans cette Note.† Relativement aux équations du type hyperbolique, l'utilité de ces méthodes est d'une double nature. Elles permettent d'abord de faire la recherche des intégrales en supposant les équations différentielles définies seulement pour les valeurs réelles des

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*Nous reproduisons ici une note insérée par M. Picard à la fin du tome IV de la Théorie des Surfaces de M. Darboux (Note de la Rédaction).

† Relativement aux théorèmes généraux relatifs à ce cas, nous énoncerons seulement la proposition suivante (Journal de l'École Polytechnique, 1890). Soit l'équation linéaire

$$A\frac{\partial^2 z}{\partial x^2}+2B\frac{\partial^2 z}{\partial x\,\partial y}+C\frac{\partial^2 z}{\partial y^2}+D\frac{\partial z}{\partial x}+E\frac{\partial z}{\partial y}+Fz=0,$$

où les coefficients sont des fonctions *analytiques* des deux variables réelles x et y : *toute intégrale de cette équation bien déterminée et continue ainsi que ses dérivées partielles des deux premiers ordres dans une région du plan pour laquelle*

$$B^2-AC<0$$

est une fonction analytique de x et y. Il est clair qu'il peut en être autrement dans une région où B^2-AC serait positif. On trouvera un théorème plus général (Comptes Rendus, Juillet 1895).

variables, et en faisant ainsi le minimum d'hypothèses sur ces équations ; c'est là un point d'un certain intérêt philosophique.

Une conséquence pratique en découle ; on obtient, en général, pour les intégrales un champ de détermination plus étendu qu'avec les méthodes fondées sur l'emploi de fonctions majorantes quand ces méthodes sont applicables.

1. Rappelons d'abord, sans y insister, les résultats relatifs à une équation ordinaire du premier ordre

$$\frac{dy}{dx} = f(x, y).$$

Si $f(x, y)$ est une fonction réelle et continue des deux variables réelles x et y, quand celles-ci varient respectivement dans les intervalles

$$(x_0 - a, x_0 + a), \quad (y_0 - b, y_0 + b),$$

et si, de plus, il existe une constante positive k telle que

$$|f(x, y + \Delta y) - f(x, y)| < k |\Delta y|,$$

x, y et $y + \Delta y$ étant les intervalles indiqués, et qu'enfin M désigne le maximum de la valeur absolue de $f(x, y)$ dans ces mêmes intervalles, les approximations successives donnent l'intégrale de l'équation prenant pour $x = x_0$ la valeur y_0, sous forme de série convergente dans l'intervalle $(x_0 - \rho, x_0 + \rho)$, en désignant par ρ la plus petite des deux quantités*

$$a \text{ et } \frac{b}{M}. \tag{1}$$

M. E. Lindelöf, qui a très heureusement approfondi cette question (Journal de Math., 1894), a même indiqué un autre champ de convergence qui peut quelquefois être plus étendu que le précédent. Désignons par M_0 la plus grande valeur absolue de $f(x, y_0)$, quand x varie de $x_0 - a$ à $x_0 + a$; un champ de con-

*Nous avons supposé la fonction $f(x, y)$ définie seulement pour les valeurs réelles de x et y. Dans le cas où $f(x, y)$ est une fonction analytique de x et y, holomorphe dans les cercles de rayons a et b tracés respectivement autour des points x_0 et y_0, et en désignant par M le module maximum de f dans ces cercles, les approximations successives permettent d'obtenir l'intégrale prenant pour $x = x_0$ la valeur y_0 sous forme de série convergente dans un cercle de rayon ρ autour de x_0 (en désignant par ρ la même quantité que ci-dessus). La méthode des fonctions majorantes donne un champ de convergence moins étendu.

vergence assurée est l'intervalle $(x_0 - \rho', x_0 + \rho')$, en désignant par ρ' la plus petite des deux quantités

$$a \quad \text{et} \quad \frac{1}{k} \log\left(1 + \frac{kb}{M_0}\right), \tag{2}$$

et ρ' peut dans certains cas dépasser ρ.

Il n'est pas sans intérêt de rappeler que la première méthode de Cauchy, telle que nous la connaissons par les leçons qu'a rédigées M. Moigno, et qui a été depuis reprise par M. Lipschitz, méthode dont le principe est de considérer l'équation différentielle comme une équation aux différences, définissait précisément l'intégrale dans l'intervalle correspondant à (1).

2. Considérons maintenant une équation aux dérivées partielles de la forme

$$\frac{\partial^2 z}{\partial x \partial y} = F\left(z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, x, y\right).$$

Les approximations successives permettent, entre autres problèmes, de former l'intégrale d'une telle équation se réduisant, pour $x = x_0$, à une fonction donnée de y, et pour $y = y_0$ à une fonction donnée de x. Je prendrai d'abord le cas de l'équation linéaire

$$\frac{\partial^2 z}{\partial x \partial y} = a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz,$$

où a, b, c sont des fonctions des deux variables réelles x et y. Nous les supposerons continues à l'intérieur et sur le périmètre d'un rectangle R de côtés α et β parallèles aux axes et dont (x_0, y_0) sera le sommet de moindres abscisse et ordonnée. On veut trouver l'intégrale de cette équation se réduisant pour $y = y_0$ à $\phi(x)$ et pour $x = x_0$ à $\psi(y)$. La fonction $\phi(x)$ est continue de x_0 à $x_0 + \alpha$, et $\psi(y)$ est continue de y_0 à $y_0 + \beta$; on a, bien entendu, $\phi(x_0) = \psi(y_0)$ et les deux fonctions ϕ et ψ ont des dérivées premières continues.

Envisageons, en premier lieu, l'équation

$$\frac{\partial^2 z}{\partial x \partial y} = f(x, y),$$

où $f(x, y)$ est une fonction donnée. La fonction

$$z = \int_{x_0}^{x}\int_{y_0}^{y} f(x, y)\, dx\, dy$$

est l'intégrale de cette équation s'annulant, pour $x = x_0$, quel que soit y, et pour $y = y_0$ quel que soit x. Ceci posé, nous formons les équations successives

$$\frac{\partial^2 z_1}{\partial x \partial y} = 0,$$
$$\frac{\partial^2 z_2}{\partial x \partial y} = a\frac{\partial z_1}{\partial x} + b\frac{\partial z_1}{\partial y} + cz_1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\frac{\partial^2 z_n}{\partial x \partial y} = a\frac{\partial z_{n-1}}{\partial x} + b\frac{\partial z_{n-1}}{\partial y} + cz_{n-1}.$$

On intégrera la première équation en cherchant son intégrale z_1 se réduisant à $\phi(x)$ pour $y = y_0$ et à $\psi(y)$ pour $x = x_0$, intégrale qui est visiblement

$$z_1 = \phi(x) + \psi(y) - \phi(x_0).$$

Pour toutes les autres fonctions $z_n \,(n > 1)$, elles sont supposées se réduire à zéro pour $x = x_0$ quel que soit y, et pour $y = y_0$ quel que soit x.

Nous allons montrer dans un moment que *les séries*

$$z_1 + z_2 \quad + \ldots\ldots + z_n \quad + \ldots\ldots,$$
$$\frac{\partial z_1}{\partial x} + \frac{\partial z_2}{\partial x} + \ldots\ldots + \frac{\partial z_n}{\partial x} + \ldots\ldots,$$
$$\frac{\partial z_1}{\partial y} + \frac{\partial z_2}{\partial y} + \ldots\ldots + \frac{\partial z_n}{\partial y} + \ldots\ldots,$$

sont uniformément convergentes dans le rectangle R. Ce point admis, on voit sans peine que *la fonction*

$$Z = z_1 + z_2 + \ldots\ldots + z_n + \ldots\ldots$$

est l'intégrale cherchée. On tire, en effet, des équations précédentes

$$z_1 + z_2 + \ldots\ldots + z_n = \phi(x) + \psi(y) - \phi(x_0) + \int_{x_0}^{x}\int_{y_0}^{y}\Big[a\frac{\partial(z_1 + \ldots\ldots + z_{n-1})}{\partial x}$$
$$+ b\frac{\partial(z_1 + \ldots\ldots + z_{n-1})}{\partial y} + c(z_1 + \ldots\ldots + z_{n-1})\Big]\,dx\,dy,$$

d'où l'on conclut à la limite, en s'appuyant sur la convergence uniforme des séries écrites plus haut,

$$Z = \phi(x) + \psi(y) - \phi(x_0) + \int_{x_0}^{x}\int^{y}\Big[a\frac{\partial Z}{\partial x} + b\frac{\partial Z}{\partial y} + cZ\Big]\,dx\,dy$$

et enfin

$$\frac{\partial^2 Z}{\partial x \partial y} = a\frac{\partial Z}{\partial x} + b\frac{\partial Z}{\partial y} + cZ.$$

Abordons donc la question de convergence. Je désigne par M le maximum de

$$\left| a\frac{\partial z_1}{\partial x} + b\frac{\partial z_1}{\partial y} + cz_1 \right|,$$

dans R, et par k le module maximum de a, b, c dans ce même rectangle, et je considère le système

$$\begin{aligned}
\frac{\partial^2 u_1}{\partial x\,\partial y} &= \mathrm{M},\\
\frac{\partial^2 u_2}{\partial x\,\partial y} &= k\frac{\partial u_1}{\partial x} + k\frac{\partial u_1}{\partial y} + ku_1,\\
&\ldots\ldots\ldots\ldots\ldots\\
\frac{\partial^2 u_n}{\partial x\,\partial y} &= k\frac{\partial u_{n-1}}{\partial x} + k\frac{\partial u_{n-1}}{\partial y} + ku_{n-1},
\end{aligned}$$

tous les u s'annulant pour $x = x_0$ quel que soit y, et pour $y = y_0$ quel que soit x.

Si nous prouvons la convergence uniforme de la série

$$u_1 + u_2 + \ldots. + u_n + \ldots., \tag{3}$$

la convergence de la série des z en résultera immédiatement, car $|z_n| < |u_{n-1}|$. Or, soit

$$u_n = k^{n-1}\mathrm{U}_n,$$

on aura

$$\begin{aligned}
\frac{\partial^2 \mathrm{U}_1}{\partial x\,\partial y} &= \mathrm{M},\\
\frac{\partial^2 \mathrm{U}_2}{\partial x\,\partial y} &= \frac{\partial \mathrm{U}_1}{\partial x} + \frac{\partial \mathrm{U}_1}{\partial y} + \mathrm{U}_1.\\
&\ldots\ldots\ldots\ldots\ldots\\
\frac{\partial^2 \mathrm{U}_n}{\partial x\,\partial y} &= \frac{\partial \mathrm{U}_{n-1}}{\partial x} + \frac{\partial u_{n-1}}{\partial y} + \mathrm{U}_{n-1}.
\end{aligned}$$

Si la série des u est convergente, la série

$$\mathrm{U}_1 + k\mathrm{U}_2 + \ldots. + k^n\mathrm{U}_{n+1} + \ldots. \tag{4}$$

représentera l'intégrale de l'équation

$$\frac{\partial^2 \mathrm{U}}{\partial x\,\partial y} = k\frac{\partial \mathrm{U}}{\partial x} + k\frac{\partial \mathrm{U}}{\partial y} + k\mathrm{U} + \mathrm{M}, \tag{5}$$

s'annulant pour $x = x_0$, quel que soit y, et pour $y = y_0$, quel que soit x. Or si nous montrons que, pour l'équation précédente, l'intégrale satisfaisant à ces conditions initiales, est une fonction holomorphe de k pour toute valeur de k, la convergence de la série (3) sera établie, car cette intégrale devra nécessairement avoir la forme (4).

Or l'équation (5) est facile à discuter. Prenant $x_0 = y_0 = 0$, nous poserons

$$U = e^{k(x+y)} V.$$

L'équation (5) devient

$$\frac{\partial^2 V}{\partial x \partial y} = (k^2 + k) V + M e^{-k(x+y)}, \tag{6}$$

L'application des approximations successives à cette dernière équation est immédiate. On a à considérer les équations

$$\frac{\partial^2 V_0}{\partial x \partial y} = M e^{-k(x+y)},$$
$$\frac{\partial^2 V_1}{\partial x \partial y} = (k^2 + k) V_0,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\frac{\partial^2 V_n}{\partial x \partial y} = (k^2 + k) V_{n-1},$$

tous les V s'annulant pour $x = 0$, ainsi que pour $y = 0$. En désignant par N la valeur absolue maxima de V_0 dans R, on aura

$$|V_n| < \frac{(k^2 + k)^n x^n y^n}{(1.2 \ldots . n)^2} N,$$

d'où l'on déduit de suite la convergence de la série

$$V_0 + V_1 + \ldots . + V_n + \ldots . ,$$

qui représente l'intégrale cherchée de l'équation (6). La méthode des approximations successives donne donc pour l'équation (6) une série convergente, quand (x, y) est dans R. Chacun des termes de cette série est une fonction holomorphe de k, et la série converge uniformément, quel que soit k, dans un domaine fini quelconque du plan de cette variable. L'intégrale V de (6) est donc une fonction *entière* de k, et il en est alors de même de l'intégrale U de (5), comme nous voulions l'établir. Les mêmes raisonnements sont valables pour les séries formées avec les dérivées partielles du premier ordre.

3. Passons au cas de l'équation non linéaire

$$\frac{\partial^2 z}{\partial x \partial y} = F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right).$$

Nous abrégerons l'exposition, sans diminuer la généralité, en supposant que $z = 0$ pour $x = 0$, et aussi pour $y = 0$. Il suffit évidemment pour cela de rem-

placer z par $z+[\phi(x)+\psi(y)-\phi(0)]$. Ceci posé, nous admettons que la fonction

$$F(x, y, z, u, v)$$

est continue quand (x, y) est dans R, quand z varie entre $-a$ et $+a$, et que u et v varient entre $-b$ et $+b$. De plus, pour x, y, z, u et v dans ces intervalles, on a

$$|F(x, y, z', u', v')-F(x, y, z, u, v)|<k_1|z'-z|+k_2|u'-u|+k_3|v'-v,$$

les k étant des constantes positives. Soit enfin M le maximum de la valeur absolue de F dans la région où cette fonction est définie.

On considère les équations successives

$$\frac{\partial^2 z_1}{\partial x\,\partial y}=F(x, y, 0, 0, 0),$$
$$\frac{\partial^2 z_2}{\partial x\,\partial y}=F\left(x, y, z_1, \frac{\partial z_1}{\partial x}, \frac{\partial z_1}{\partial y}\right),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\frac{\partial^2 z_n}{\partial x\,\partial y}=F\left(x, y, z_{n-1}, \frac{\partial z_{n-1}}{\partial x}, \frac{\partial z_{n-1}}{\partial y}\right),$$

les z s'annulant tous pour $x=x_0$, quel que soit y, et pour $y=y_0$ quel que soit x. On sera assuré que

$$z_n, \frac{\partial z_n}{\partial x}, \frac{\partial z_n}{\partial y}$$

restent compris dans les limites indiquées, si (x, y) est à l'intérieur d'un rectangle compris dans R, ayant pour sommet (x_0, y_0), et dont les côtés ρ et ρ' satisfont aux inégalités

$$M\rho\rho'<a, \quad M\rho<b, \quad M\rho'<b.$$

Nous supposerons d'ailleurs que ρ et ρ' sont au plus égaux aux côtés α et β du rectangle R.

Dans ces conditions la série

$$z_1+z_2+\cdots+z_n+\cdots$$

représentera l'intégrale cherchée. On est, en effet, ramené immédiatement au cas de l'équation linéaire, en considérant les équations

$$\frac{\partial^2 z_1}{\partial x\,\partial y}=F(x, y, 0, 0, 0),$$
$$\frac{\partial^2 (z_2-z_1)}{\partial x\,\partial y}=F\left(x, y, z_1, \frac{\partial z_1}{\partial x}, \frac{\partial z_1}{\partial y}\right)-F(x, y, 0, 0, 0),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\frac{\partial^2 (z_n-z_{n-1})}{\partial x\,\partial y}=F\left(x, y, z_{n-1}, \frac{\partial z_{n-1}}{\partial x}, \frac{\partial z_{n-1}}{\partial y}\right)-F\left(x, y, z_{n-2}, \frac{\partial z_{n-2}}{\partial x}, \frac{\partial z_{n-2}}{\partial y}\right),$$

et en leur substituant les équations linéaires

$$\frac{\partial^2 u_n}{\partial x \partial y} = k_1 u_{n-1} + k_2 \frac{\partial u_{n-1}}{\partial x} + k_3 \frac{\partial u_{n-1}}{\partial y}.$$

La convergence de la suite déduite de ces dernières équations entraîne immédiatement la convergence de la série des z dans le rectangle (ρ, ρ'), et le problème est par suite résolu. *L'intégrale est déterminée dans le rectangle* (ρ, ρ').

4. Il est remarquable que, dans la question précédente, les limites trouvées pour ρ et ρ' ne dépendent pas des constantes k. Il faut cependant que l'on soit assuré de l'existence de ces constantes pour que le raisonnement soit valable. Un cas intéressant est celui où la fonction

$$F(x, y, z, u, v)$$

serait déterminée et continue pour toute valeur réelle de z, u et v [le point (x, y) étant dans le rectangle R] et où cette fonction aurait des dérivées premières

$$\frac{\partial F}{\partial z}, \frac{\partial F}{\partial u}, \frac{\partial F}{\partial v},$$

restant en valeur absolue moindre qu'un nombre fixe dans les mêmes conditions.

Nous n'aurons pas alors à nous préoccuper des inégalités (7), puisque la fonction F est déterminée pour toute valeur de z, u, v; par suite, dans ce cas, *la série représentant l'intégrale convergera dans* R.

Ainsi, par exemple, l'équation

$$\frac{\partial^2 z}{\partial x \partial y} = a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} + c \sin z,$$

où a, b, c sont des fonctions continues de x et y dans le rectangle R, admettra ce rectangle même comme champ de convergence pour la série donnée par les approximations successives. On sait que l'équation

$$\frac{\partial^2 z}{\partial x \partial y} = \sin z$$

se rencontre dans la théorie des surfaces à courbure constante, et, dans ses leçons de *Géométrie différentielle,* M. Bianchi s'est servi des approximations successives appliquées à cette équation pour traiter un intéressant problème de Géométrie.

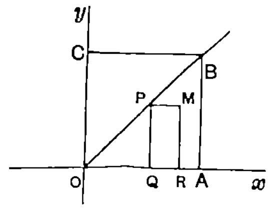
5. Bien d'autres problèmes concernant les équations précédentes pourraient être traités par une autre voie analogue. Pour indiquer au moins un nouvel

exemple reprenons l'équation linéaire

$$\frac{\partial^2 z}{\partial x \, \partial y} = a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} + cz,$$

et construisons sur Ox et Oy (Fig. 89) le carré OABC de côtés $OA = OC = \alpha$,

Fig. 89.



que nous désignerons par R. On se donne la valeur d'une intégrale z sur OA et sur OB ; on aura ainsi

$$z = f(x) \text{ pour } y = 0,$$
$$z = \phi(x) \text{ pour } y = x;$$

$f(x)$ et $\phi(x)$ sont deux fonctions continues ainsi que leurs dérivées du premier ordre ; elles sont définies de $x = 0$ à $x = \alpha$, et l'on a, bien entendu, $f(0) = \phi(0)$. L'intégrale de l'équation

$$\frac{\partial^2 z}{\partial x \, \partial y} = 0,$$

satisfaisant à ces conditions initiales, sera évidemment

$$z = f(x) + \phi(y) - f(y).$$

Ensuite, en désignant par $P(x, y)$ une fonction donnée de x et de y dans le carré R, l'intégrale de l'équation

$$\frac{\partial^2 u}{\partial x \, \partial y} = P(x, y),$$

s'annulant sur OA et sur OB, sera

$$u = \int_0^y d\eta \int_y^x P(\xi, \eta) \, d\xi;$$

le champ d'intégration est le rectangle MPQR, en désignant par M le point (x, y).

Formons alors, comme précédemment, le système

$$\frac{\partial^2 z_1}{\partial x \, \partial y} = 0,$$
$$\frac{\partial^2 z_2}{\partial x \, \partial y} = a \frac{\partial z_1}{\partial x} + b \frac{\partial z_1}{\partial y} + cz_1,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\frac{\partial^2 z_n}{\partial x \, \partial y} = a \frac{\partial z_{n-1}}{\partial x} + b \frac{\partial z_{n-1}}{\partial y} + cz_{n-1}.$$

On intègre la première avec les conditions

$$z_1 = f(x) \text{ pour } y = 0 \text{ et } z_1 = \phi(x) \text{ pour } y = x,$$

et pour n supérieur à *un*, on prend

$$z_n = 0 \text{ pour } y = 0 \text{ et } z_n = 0 \text{ pour } y = x.$$

Des considérations analogues à celles que nous avons employées ci-dessus permettent aisément d'établir que la série

$$z_1 + z_2 + \dots + z_n + \dots$$

converge uniformément dans R, et représente l'intégrale cherchée.

6. Comme exemple d'équations d'ordre supérieur au second, pour lesquelles s'appliquent sans difficultés les méthodes précédentes, je citerai les équations suivantes étudiées à ce point de vue par M. Delassus dans un des chapitres de son intéressante thèse (*voir* aussi Comptes rendus, 1893). Ce sont les équations d'ordre n de la forme

$$\sum_{i,k} A_{ik} \frac{\partial^{i+k} z}{\partial x^i \partial y^k} = 0,$$

avec les conditions suivantes:

$$\begin{matrix} i = 0, 1, \dots p \\ k = 0, 1, \dots q \end{matrix} \quad (p + q = n, \ pq \neq 0)$$

et en supposant $A_{pq} = 1$.

7. Je voudrais maintenant considérer des équations pour lesquelles on ne puisse appliquer la méthode précédente d'approximations. Il n'est pas difficile de trouver de tels exemples, nous n'avons qu'à prendre l'équation du premier ordre

$$\frac{\partial z}{\partial x} = a(x, y) \frac{\partial z}{\partial y}. \tag{8}$$

Supposons qu'on veuille trouver l'intégrale de cette équation se réduisant, pour $x = x_0$, à une fonction donnée F (y). On peut former les équations suivantes

$$\frac{\partial z_1}{\partial x} = 0,$$

$$\frac{\partial z_2}{\partial x} = a(x, y) \frac{\partial z_1}{\partial y},$$

$$\dots\dots\dots\dots\dots$$

$$\frac{\partial z_n}{\partial x} = a(x, y) \frac{\partial z_{n-1}}{\partial y},$$

z_1 prenant pour $x = x_0$ la valeur F (y), et les autres z s'annulant identiquement pour cette valeur de x. Mais on voit que l'on ne pourra former les fonctions $z_2, \ldots, z_n, \ldots$ que si F (y) et $a\,(x, y)$ ont des dérivées partielles de tout ordre par rapport à y, et la convergence du développement ne peut être établie que si l'on suppose que F(y) et $a\,(x, y)$ sont des fonctions analytiques. *Il semble donc qu'on ne puisse établir l'existence des intégrales de l'équation* (8) *qu'en admettant que* $a\,(x, y)$ *est analytique.* Quoique la question n'ait qu'un intérêt théorique, elle vaut peut-être la peine d'être examinée.

Reprenons d'abord, à cet effet, l'étude de l'équation différentielle ordinaire

$$\frac{dy}{dx} = f(x, y),$$

et plaçons-nous dans les hypothèses du n° 1. Nous avons trouvé une intégrale prenant pour $x = x_0$ la valeur y_0, soit

$$y = \mathrm{F}(x, x_0, y_0),$$

en mettant en évidence toutes les quantités dont dépend F. La fonction F est une fonction continue de x, x_0 et y_0; elle a une dérivée première par rapport à x, mais toute la difficulté de la question qui nous occupe est de savoir si cette fonction a une dérivée partielle du premier ordre par rapport à y_0. Or les approximations successives nous conduisent à la suite de fonctions

$$y_1, y_2, \ldots, y_n, \ldots,$$

se calculant de proche en proche au moyen des formules

$$\begin{aligned}
y_1 &= y_0 + \int_{x_0}^{x} f(x, y_0)\, dx, \\
y_2 &= y_0 + \int_{x_0}^{x} f(x, y_1)\, dx, \\
&\cdots\cdots\cdots\cdots\cdots \\
y_n &= y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx,
\end{aligned}$$

et F(x, x_0, y_0) est la limite de y_n. On peut calculer de proche en proche les dérivées partielles

$$\frac{\partial y_1}{\partial y_0}, \frac{\partial y_2}{\partial y_0}, \ldots, \frac{\partial y_n}{\partial y_0},$$

si l'on admet seulement que $f(x, y)$ a une dérivée partielle du premier ordre par rapport à y. Il est donc bien vraisemblable que F aura une dérivée partielle du

premier ordre par rapport à y_0. Nous le démontrerons élégamment sans calculs en rattachant la question à un problème traité plus haut; on va supposer que $f(x, y)$ a des dérivées partielles des deux premiers ordres par rapport à y. [Il suffirait même d'admettre que la dérivée $\frac{\partial f}{\partial y}$, sans avoir de dérivée par rapport à y, jouit de la propriété dont jouissait la fonction appelée $f(x, y)$ au n° 1.]

Je considérerai, dans ce qui va suivre, x_0 comme une constante numérique et β désignera une seconde quantité numérique. J'envisage l'équation aux dérivées partielles

$$\frac{\partial^2 y}{\partial x \, \partial y_0} = \frac{\partial f}{\partial y} \frac{\partial y}{\partial y_0}, \tag{9}$$

définissant une fonction y des deux variables x et y_0. D'après ce qui a été vu au n° 3, nous pouvons l'intégrer en prenant les conditions initiales suivantes:

$$\begin{aligned} y &= y_0 & &\text{pour} \quad x = x_0, \\ y &= \mathrm{F}(x, x_0, \beta) & &\text{pour} \quad y_0 = \beta. \end{aligned}$$

L'intégrale y de l'équation (9) sera alors complètement déterminée. Or, on déduit de cette équation

$$\frac{\partial y}{\partial x} = f(x, y) + \mathrm{P}(x),$$

$\mathrm{P}(x)$ ne dépendant pas de y_0. Or, pour $y_0 = \beta$, on a

$$\frac{\partial y}{\partial x} = f(x, y),$$

puisque, pour $y_0 = \beta$, y est l'intégrale de l'équation $\frac{dy}{dx} = f(x, y)$, qui prend, pour $x = x_0$, la valeur β. La fonction $\mathrm{P}(x)$ est donc identiquement nulle, et l'intégrale

$$y(x, y_0),$$

que nous venons d'obtenir, est l'intégrale de l'équation $\frac{dy}{dx} = f(x, y)$ prenant pour $x = x_0$ la valeur y_0; elle est identique à la fonction $\mathrm{F}(x, x_0, y_0)$, et celle-ci a, par suite, une dérivée du premier ordre par rapport à y_0.

On démontrera d'une manière analogue que $\mathrm{F}(x, x_0, y_0)$ a une dérivée du premier ordre par rapport à x_0. On regardera y_0 comme une constante numérique,

et soit α une seconde quantité numérique. Nous formons l'équation

$$\frac{\partial^2 y}{\partial x \, \partial x_0} = \frac{\partial f}{\partial y} \frac{\partial y}{\partial x_0}, \tag{10}$$

en l'intégrant avec les conditions initiales

$$\begin{array}{ll} y = y_0 & \text{pour} \quad x = x_0, \\ y = \mathrm{F}(x, \alpha, y_0) & \text{pour} \quad x_0 = \alpha, \end{array}$$

ce qui correspond au cas étudié (n° 5). On déduit de l'équation (10)

$$\frac{\partial y}{\partial x} = f(x, y) + \mathrm{Q}(x).$$

$\mathrm{Q}(x)$ ne dépendant pas de x_0; on voit que $\mathrm{Q}(x)$ est nul, en faisant dans cette relation $x_0 = \alpha$, et l'on termine comme plus haut.

Ainsi, la fonction $\mathrm{F}(x, x_0, y_0)$ a des dérivées partielles du premier ordre par rapport à x_0 et à y_0. Or, la relation

$$y = \mathrm{F}(x, x_0, y_0)$$

peut manifestement s'écrire

$$y_0 = \mathrm{F}(x_0, x, y),$$

puisque l'intégrale qui, pour la valeur x de la variable, prend la valeur y aura en x_0 la valeur y_0. D'après ce qui précède, $\mathrm{F}(x_0, x, y)$ est une fonction continue de x et y, et elle a des dérivées partielles du premier ordre elles-mêmes continues. Désignons cette fonction par

$$\mathrm{F}(x, y),$$

en n'écrivant plus la constante x_0: nous aurons l'intégrale générale de l'équation $\frac{dy}{dx} = f(x, y)$ sous la forme

$$\mathrm{F}(x, y) = \text{const.},$$

et F satisfera à l'équation aux dérivées partielles

$$\frac{\partial \mathrm{F}}{\partial x} + f(x, y) \frac{\partial \mathrm{F}}{\partial y} = 0.$$

Nous avons donc établi l'existence d'une intégrale de cette équation, et par suite de toutes les intégrales, en supposant seulement que $f(x, y)$ est continue et a des dérivées partielles des deux premiers ordres par rapport à y.

Ainsi, comme application, on peut établir l'existence d'un facteur intégrant pour l'expression

$$dy + \mathrm{P}(x, y)\,dx,$$

en supposant seulement que la fonction $\mathrm{P}(x, y)$ est continue et a des dérivées partielles des deux premiers ordres par rapport à y.

8. Tout ce que nous venons de dire subsistera évidemment si les fonctions considérées, au lieu d'être réelles, sont des fonctions complexes des deux variables réelles x et y. En particulier, le théorème relatif au facteur intégrant qui vient d'être énoncé s'applique aussi bien si l'on a

$$\mathrm{P}(x, y) = p(x, y) + iq(x, y),$$

p et q étant des fonctions réelles de x et y, jouissant des propriétés indiquées.

Une application, qui offre quelque intérêt, se présente immédiatement. C'est une proposition élémentaire, qu'une surface *analytique* peut être représentée sur un plan, de manière qu'il y ait conservation des angles : on a ainsi une carte géographique de la surface. La démonstration bien connue de ce théorème s'appuie essentiellement sur ce que la surface est analytique ; elle revient à la recherche d'un facteur intégrant. Avec l'extension donnée à cette dernière recherche, nous n'avons plus besoin d'admettre que la surface est analytique. Soit une surface pour laquelle le carré de l'élément linéaire se mette sous la forme

$$ds^2 = \mathrm{E}\,dx^2 + 2\mathrm{F}\,dx\,dy + \mathrm{G}\,dy^2.$$

On peut, d'après ce que nous venons de dire, démontrer la possibilité de faire la carte de cette surface sur un plan, si les trois coefficients E, F, G sont des fonctions continues de x et y, ayant des dérivées partielles des deux premiers ordres par rapport à y. Il suffit même de supposer que les trois dérivées du premier ordre

$$\frac{\partial \mathrm{E}}{\partial y}, \quad \frac{\partial \mathrm{F}}{\partial y}, \quad \frac{\partial \mathrm{G}}{\partial y},$$

jouissent de la propriété admise pour la fonction $f(x, y)$ au n° 1.

Ces conditions, un peu dissymétriques, sont suffisantes ; elles ne sont sans doute pas toutes nécessaires, mais, pour s'en affranchir, il faudrait trouver un autre mode de démonstration pour l'existence du facteur intégrant.

Focal Surfaces of the Congruences of Tangents to a given Surface.

By A. Pell.

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§1.

ider a surface (Σ) referred to its lines of curvature u and v. For 'e shall call the u-line and the v-line those lines for which $v =$ const. const. respectively. The axis of x is tangent to the u-line and drawn ection of the increasing arc; the axis of y is tangent to the v-line, and ion is such that a rotation around the axis of z, bringing the x-axis into ce with the y-axis, is represented by a line directed along the positive he z-axis. The direction of the z-axis is that of the normal to the

g Darboux's notation, we have for (Σ) the following formulæ:*

$$
\left.
\begin{array}{l}
\xi = \sqrt{E}, \ \eta = 0, \ \xi_1 = 0, \ \eta_1 = \sqrt{G}, \ p = 0, \\
{}_1 = 0, \quad r = -\frac{1}{\sqrt{G}} \cdot \frac{\partial \sqrt{E}}{\partial v}, \ r_1 = \frac{1}{\sqrt{E}} \cdot \frac{\partial \sqrt{G}}{\partial u}, \ \zeta = 0, \ \zeta_1 = 0. \\
\frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u} = -q p_1, \ \frac{\partial p_1}{\partial u} = -q r_1, \ \frac{\partial q}{\partial v} = r p_1, \\
\frac{\partial}{\partial u}\left(\frac{1}{q} \cdot \frac{\partial p_1}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{p_1} \cdot \frac{\partial q}{\partial v}\right) + q p_1 = 0.
\end{array}
\right\} \quad (1)
$$

, ζ and ξ_1, η_1, ζ_1 are the components of the velocity of translation of of the movable axes relatively to these axes when u or v vary respec-

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* Darboux, "Théorie générale des surfaces," vol. II, p. 386.

tively, and p, q, r, p_1, q_1, r_1 are the rotations around the axes, the first three when only u varies, the last three when only v varies.

Then calling ρ_1 and ρ_2 the principal radii of curvature of the u-line and the v-line respectively, we have

$$\rho_1 = -\frac{\sqrt{E}}{q_1}, \quad \rho_2 = \frac{\sqrt{G}}{p_1}. \tag{2}$$

Further, let h_1 and h_2 be the principal radii of curvature of the u- and the v-lines; τ_1 and τ_2 their radii of torsion; R_2 and R_1 the radii of geodesic curvatures, directed along the positive parts of the y and the x axes respectively. For these radii we have

$$r = \frac{\sqrt{E}}{R_2}, \quad r_1 = -\frac{\sqrt{G}}{R_1}, \tag{3}$$

and since

$$\frac{\sqrt{E}}{t_1} = -p = 0; \quad \frac{\sqrt{G}}{t_2} = -q_1 = 0, \tag{4}$$

where t_1 and t_2 are the radii of geodesic torsions of the u- and the v-lines respectively, we have

$$\frac{1}{\tau_1} = \frac{1}{\sqrt{E}} \cdot \frac{\partial \varpi_1}{\partial u}, \quad \frac{1}{\tau_2} = \frac{1}{\sqrt{G}} \cdot \frac{\partial \varpi_2}{\partial v}. \tag{5}$$

Here ϖ_1 and ϖ_2 are the angles between the radii of principal curvatures of the surface (Σ) and the segments h_1 and h_2.

We have

$$\left.\begin{aligned} h_1 &= \rho_1 \cos \varpi_1, & h_2 &= \rho_2 \cos \varpi_2, \\ h_1 &= R_2 \sin \varpi_1, & h_2 &= R_1 \sin \varpi_2, \\ \frac{1}{h_1^2} &= \frac{1}{R_2^2} + \frac{1}{\rho_1^2}, & \frac{1}{h_2^2} &= \frac{1}{R_1^2} + \frac{1}{\rho_2^2}. \end{aligned}\right\} \tag{6}$$

§2.

Our consideration of the focal surfaces of the congruences of the tangents to the lines of curvature of the surface (Σ), is based on the following two theorems given by Darboux* and Koenigs.†

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* L. c. vol. III, p. 121.

† G. Koenigs, "Sur les propriétés infinitésimal de l'espace réglé," Ann. de l'école normale, etc. 1882, p. 248.

orem I.—The locus of the centers of geodesic curvature of lines of cu ny surface is the edge of regression of the developable surface, genera ngent planes of the surface at all points of the lines of curvature.

theorem shows that the distances between the point M of (Σ) and nding points M' of the focal surfaces under discussion are: for the fam t., R_1, and for the family $u=$ const., R_2.

orem II.—The edges of regression of the developable surfaces of a c form two families of curves on the focal surfaces (say S_A and S_B cor to the focal surfaces A and B), the osculating planes of which o the surfaces B and A respectively, and the points of contact desc surfaces two families of conjugate lines S_A and S_B.

theorem states that the osculating plane of u-line is tangent to ace (S_1) and that of the v-line is tangent to (S_2); and further, that to these surfaces are parallel to the binormals of the u- and v-li ely.

ce the study of the focal surface (S_1) of the congruence of the tange line can be reduced to the consideration of the motion of a trihed t the center of geodesic curvature of the v-line by the tangent to e line parallel to the segment h_1 and the parallel to the binormal, dra ble direction. The motion of this trihedron we are going to determ t of the trihedron formed by the tangents to the u- and the v-lines al to the surface (Σ).

study of the motion of the trihedron formed at the center of geod e of the u-line by the tangent to the v-line, the parallel to h_2 and o the binormal of the v-line, will give us means of discussing the fc S_2).

us call the trihedron connected with the surface (Σ), T. Conside y, z) connected with this trihedron. The projections δx, δy, δz on n infinitely small displacement of this point are:

$$\left.\begin{aligned} \delta x &= dx + \sqrt{E}\,du + zq\,du - y\,(r du + r_1 dv)\,, \\ \delta y &= dy + \sqrt{G}\,dv + x\,(r du + r_1 dv) - zp_1\,dv\,, \\ \delta z &= dz + yp_1\,dv - xq\,du\,. \end{aligned}\right\}$$

now that this point coincide in the first place with the center of geod

curvature of the v-line ($x = R_1$, $y = 0$, $z = 0$), and in the second place with that of the u-line ($x = 0$, $y = R_2$, $z = 0$). We know that

$$\sqrt{E}\,du = ds_1, \quad \sqrt{G}\,dv = ds_2,$$

s_1 and s_2 being the lengths of the arcs of the u- and the v-lines respectively. The above displacements take the form

$$\begin{array}{ll} \text{for } (S_1), & \text{for } (S_2), \\ \delta x_1 = \quad dR_1 + ds_1, & \delta x_2 = - \quad R_2 r_1\, dv \quad , \\ \delta y_1 = \quad R_1\, r du \quad , & \delta y_2 = \quad dR_2 + ds_2 , \\ \delta z_1 = - \quad R_1\, q du \quad , & \delta z_2 = \quad R_2\, p_1\, dv \ . \end{array}$$

Substituting for r, q, r_1, p_1, their values, we get

$$\begin{array}{ll} \delta x_1 = dR_1 + ds_1 , & \delta x_2 = \dfrac{R_2}{R_1}\, ds_2 \quad , \\ \delta y_1 = \dfrac{R_1}{R_2}\, ds_1 \quad , & \delta y_2 = d R_2 + ds_2 , \\ \delta z_1 = \dfrac{R_1}{\rho_1}\, ds_1 \quad , & \delta z_2 = \dfrac{R_2}{\rho_2}\, ds_2 \quad . \end{array}$$

These displacements are with respect to the trihedron (T). Codazzi's formulæ, as given in Darboux's "Leçons, etc.," require that the tangent plane to the surface contain the axes x and y. But according to theorem II, the osculating plane of the u-line is tangent to (S_1) at the corresponding point; that of the v-line is tangent to (S_2); therefore we must find the values of the projections of the above displacements on the axes of the trihedrons (T_1) and (T_2) formed as stated above. The rotations of (T_1) and (T_2) are obtained by compounding the latter and the rotations through the angles ϖ_1 and ϖ_2 around the axes x and y respectively.

Following is a table giving the direction cosines of the new axes and the old ones.

	x'	y'	z'	x''	y''	z''
x	1	0	0	$\sin \varpi_2$	0	$-\cos \varpi_2$
y	0	$\sin \varpi_1$	$\cos \varpi_1$	0	1	0
z	0	$\cos \varpi_1$	$-\sin \varpi_1$	$\cos \varpi_2$	0	$\sin \varpi_2$

', $\delta z_1'$ and $\delta x_2''$, $\delta y_2''$, $\delta z_2''$ are the displacements with respect to (T_1
ively, we have

$$\begin{aligned} &{}_1' = \delta x_1 \cos(xx') + \delta y_1 \cos(yx') + \delta z' \cos(zx) = \delta x_1, \\ &{}_1' = \delta y_1 \sin \varpi_1 + \delta z_1 \cos \varpi_1, \\ &{}_1' = \delta z_1 \cos \varpi_1 - \delta z_1 \sin \varpi_1, \end{aligned}$$

ution,

$$\left.\begin{aligned} \delta x_1' &= dR_1 + ds_1, \\ \delta y_1' &= \frac{R_1}{h_1} ds_1 \quad , \\ \delta z_1' &= 0 \quad , \end{aligned}\right\} \text{for } (S_1), \tag{8}$$

$$\left.\begin{aligned} \delta x_2'' &= \frac{R_2}{h_2} ds_2 \quad , \\ \delta y_2'' &= dR_2 + ds_2, \\ \delta z_2'' &= 0 \quad , \end{aligned}\right\} \text{for } (S_2). \tag{9}$$

ns we could have obtained at once by remarking that, fo
al surface (S_1) can be generated by the motion of the trihedro
-line, and then considering in the formulæ for the displacement
arameter. In this case we have

$$dv = 0\,; \ \xi\, du = ds_1\,; \ \eta = 0, \ \zeta = 0,$$
$$p = -\frac{1}{\tau_1}, \quad q = 0, \quad r = \frac{1}{h_1}.$$

$$du = 0, \ \eta_1\, dv = ds_2, \ \xi_1 = 0, \ \zeta_1 = 0,$$
$$p_1 = 0, \qquad q_1 = \frac{1}{\tau_2}, \ r_1 = -\frac{1}{h_2}.$$

of the displacements on the axes of x, y, z now are:

$$\left.\begin{aligned} \delta x_1' &= dx - \frac{y}{h_1} . ds_1 + ds_1, \\ \delta y_1' &= dy + \left(\frac{x}{h_1} + \frac{z}{\tau_1}\right) ds_1, \\ \delta z_1' &= dz - \frac{y}{\tau_1} ds_1 \end{aligned}\right\} \tag{10}$$

and

$$\left.\begin{aligned}\delta x_2'' &= dx + \left(\frac{z}{\tau_2} + \frac{y}{h_2}\right) ds_2,\\ \delta y_2'' &= dy + ds_2 + \frac{x}{h_2} ds_2,\\ \delta z_2'' &= dz + \frac{x}{\tau_2} ds_2.\end{aligned}\right\} \tag{11}$$

Applying these formulæ to the points $(R_1, 0, 0)$ and $(0, R_2, 0)$, we get

$$\left.\begin{aligned}\delta x_1' &= dR_1 + ds_1,\\ \delta y_1' &= \frac{R_1}{h_1} . ds_1,\\ \delta z_1' &= 0,\\ \text{and} \qquad \delta x_2'' &= \frac{R_2}{h_2} ds_2,\\ \delta y_2'' &= dR_2 + ds_2,\\ \delta z_2'' &= 0.\end{aligned}\right\} \tag{12}$$

§3.

The expressions for the linear elements of the arcs ds_1' and ds_2' of the focal surfaces (S_1) and (S_2) are:

$$\left.\begin{aligned}ds_1'^2 &= \Sigma\delta x'^2 = (dR_1 + ds_1)^2 + \frac{R_1^2}{h_1^2} ds_1^2,\\ ds_2'^2 &= \Sigma\delta x''^2 = (dR_2 + ds_2)^2 + \frac{R_2^2}{h_2^2} ds_2^2.\end{aligned}\right\} \tag{13}$$

Hence, for (S_1),

$$E_1 = \left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right)^2 + \left(\frac{R_1}{h_1}\right)^2 E,$$

$$G_1 = \left(\frac{\partial R_1}{\partial v}\right)^2,$$

$$F_1 = \frac{\partial R_1}{\partial v}\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right),$$

and for (S_2),

$$E_2 = \left(\frac{\partial R_2}{\partial u}\right)^2,$$

$$G_2 = \left(\frac{\partial R_2}{\partial v} + \sqrt{G}\right)^2 + \frac{R_2^2}{h_2^2} G,$$

$$F_2 = \frac{\partial R_2}{\partial u}\left(\frac{\partial R_2}{\partial v} + \sqrt{G}\right).$$

These formulæ have been obtained by Professor Craig, who also deduced by aid of them the theorems stated by Darboux (l. c. vol. III, p. 122), that the only surfaces whose lines of curvature have their geodesic curvatures constant, are the surfaces of revolution, the cones, the cylinders and their transformations by inversion.

The angles α_1 and α_2 between the lines u and v on the surfaces (S_1) and (S_2) are given by the following formulæ:

$$\cos\alpha_1 = \frac{\dfrac{\partial R_1}{\partial u} + \sqrt{E}}{\sqrt{E_1}}; \quad \sin\alpha_1 = \frac{R_1\sqrt{E}}{h_1\sqrt{E_1}},$$

$$\cos\alpha_2 = \frac{\dfrac{\partial R_2}{\partial v} + \sqrt{G}}{\sqrt{G_2}}; \quad \sin\alpha_2 = \frac{R_2\sqrt{G}}{h_2\sqrt{G_2}}.$$

To find the velocities of translations we use the formulæ

$$\xi_\kappa = \sqrt{E_\kappa}\,.\cos m_\kappa, \quad \eta_\kappa = \sqrt{E_\kappa}\,.\sin m_\kappa,$$
$$\xi_1^{(\kappa)} = \sqrt{G_\kappa}\,.\cos n_\kappa, \quad \eta_1^{(\kappa)} = \sqrt{G_\kappa}\,.\sin n_\kappa,$$

where

$$n_\kappa - m_\kappa = \alpha_\kappa. \qquad (\varkappa = 1,\ 2)$$

Now for (S_1) the axis of x is tangent to the v-line, hence $n_1 = 0$, $m_1 = -\alpha_1$; for (S_2) the axis of y is tangent to the u-line, i. e. $m_2 = \frac{3\pi}{2}$, so that

$$\cos m_1 = \cos\alpha_1, \quad \sin m_1 = -\sin\alpha_1, \quad \cos n_1 = 1, \quad \sin n_1 = 0;$$
$$\cos m_2 = 0, \quad \sin m_2 = -1, \quad \sin n_2 = -\cos\alpha_2, \quad \cos n_2 = \sin\alpha_2.$$

The translations are:

For (T_1),

$$\xi' = \frac{\partial R_1}{\partial u} + \sqrt{E}; \quad \eta' = -\frac{R_1}{h_1}\sqrt{E}; \quad \zeta' = 0,$$
$$\xi_1' = \frac{\partial R_1}{\partial v}; \quad \eta_1' = 0; \quad \zeta_1' = 0.$$

For (T_2),

$$\xi^{(2)} = 0; \quad \eta^{(2)} = -\frac{\partial R_2}{\partial u}; \quad \zeta^{(2)} = 0,$$
$$\xi_1^{(2)} = \frac{R_2\sqrt{G}}{h_2}; \quad \eta_1^{(2)} = -\left(\frac{\partial R_2}{\partial u} + \sqrt{G}\right); \quad \zeta_1^{(2)} = 0.$$

§4.

Returning now to the linear elements ds_1 and ds_2, we see that by making in (13)

$$R_1 + s_1 = Q_1, \quad R_2 + s_2 = Q_2,$$

we can write them as follows:

$$\left.\begin{aligned} ds_1'^2 &= dQ_1^2 + \frac{R_1^2}{h_1^2}\, ds_1^2, \\ ds_2'^2 &= dQ_2^2 + \frac{R_2^2}{h_2^2}\, ds_2^2, \end{aligned}\right\} \tag{14}$$

i. e. the lines

$$\begin{aligned} R_1 + s_1 &= \text{const.}, \\ R_2 + s_2 &= \text{const.}, \end{aligned}$$

are respectively the orthogonal trajectories of the geodesics

$$\begin{aligned} s_1 &= \text{const.}, \\ s_2 &= \text{const.} \end{aligned}$$

on the surfaces (S_1) and (S_2).

From the above expressions of $ds_1'^2$ and $ds_2'^2$ we also see that all the surfaces for which either

$$\frac{R_1}{h_1} \quad \text{or} \quad \frac{R_2}{h_2}$$

is expressed by the same function of $R_1 + s_1$ and s_1, or $R_2 + s_2$ and s_2 have the corresponding focal surfaces applicable to one another.

If, moreover, this ratio is a function of $R_1 + s_1$ or $R_2 + s_2$ only, the corresponding focal surfaces are applicable to surfaces of revolution.

Another interesting case arises when we have, say

$$h_1 = f(R_1).$$

Then taking the expression of the linear element in the form

$$ds_1'^2 = (dR_1 + ds_1)^2 + \left(\frac{R_1}{h_1}\right)^2 ds_1^2$$

we get

$$ds_1'^2 = dR_1^2 + 2dR_1\, ds_1 + \left(1 + \left(\frac{R_1}{h_1}\right)^2\right) ds_1^2,$$

then

$$1 + \left(\frac{R_1}{h_1}\right)^2 = F(R_1),$$

$$ds_1'^2 = dR_1^2 + 2dR_1 d$$

$F(R_1)$ outside of the parenthesi

$$ds_1'^2 = F(R_1)\Big[\Big(\frac{dR_1}{\sqrt{F(R_1)}}\Big)^2 + $$

ing

$$\frac{dR_1}{\sqrt{F(R_1)}} = dR_1', \text{ i. e. } R_1' = \int \frac{dR}{\sqrt{F(}}$$

$$ds_1'^2 = \Big[\frac{1}{\phi(R_1')}\Big]^2 \{dR_1'^2 + $$

$$= \Big\{\frac{1}{\phi(R_1')}\Big\}^2 [ds_1 + (\phi$$

$$\times [ds_1 + (\phi(R_1') - i$$

now

$$ds_1 + (\phi(R_1') + i\sqrt{}$$

$$ds_1 + (\phi(R_1') - i\sqrt{}$$

$$ds_1'^2 = \psi(\alpha, \beta)\, d\alpha\, d\beta$$

ows that α and β are isometric

tric lines we can find by quadr

$$\alpha = s_1 + \int [\phi(R_1') -$$

$$\beta = s_1 + \int [\phi(R_1') -$$

see that in this case the deter

educed to quadratures.

determine the function $\psi(\alpha, \beta$

$d\alpha$ and get

$$d\alpha - d\beta = 2i$$

sequently

$$\alpha - \beta = 2i$$

ves R_1' as a function of α and

ψ depends on a quadrature.

ences

$(R_1'))]$,

system of isometric
of revolution. The

g_1^2,

ly on R_1, gives us a
x, l. c. vol. III, p. 2).

esic lines

This gives us

$$c' = s_1 + \int \left[\frac{1 - \dfrac{R_1}{h_1\sqrt{1+\dfrac{R_1^2}{h_1^2}-c^2}}}{1+\dfrac{R_1^2}{h_1^2}} \right] dR_1,$$

and thus the determination of the geodesic lines on (S_1) is reduced to a quadrature.

Since $V_1 = \phi^2(R_1')$ depends exclusively on the form of the relation $h_1 = f(R_1)$, we conclude that if we have a series of surfaces (Σ) for which h_1 is the same function of R_1, the focal surfaces (S_1) will be applicable to one another and to the same surface of revolution.

Another interesting case is when

$$h_1 = f(s_1),$$

i. e. h_1 is constant along the line $s_1 =$ const. Then the focal surface is a developable surface. Because

$$ds_1'^2 = ds_1^2 + dR_1^2 + 2dR_1\,ds_1 + \frac{R_1^2}{[f(s_1)]^2}\,ds_1^2,$$

or

$$ds_1'^2 = (ds_1 + dR_1)^2 + \frac{R_1^2}{[f(s_1)]^2}\,ds_1^2.$$

Making

$$ds_1 + dR_1 = dv_1,$$

$$\int \frac{ds_1}{f(s_1)} = U_1,$$

we have

$$ds_1'^2 = dv_1^2 + (v_1 - U_1')^2\,dU_1^2,$$

where U_1' is a function of U_1 only. Further on we shall prove that (S_1) can be a developable surface only when the u-line is a plane curve. Hence we conclude that h_1 can be a function of the arc of the line of curvature only when this line is a plane curve. As an example we take a "surface moulure" defined by the equations

$$x = aU\cos\frac{v}{a} + \int V\sin\frac{v}{a}\,dv,$$

$$y = aU\sin\frac{v}{a} - \int V\cos\frac{v}{a}\,dv,$$

$$z = \int\sqrt{1 - a^2U'^2}\,du.$$

Here U and V are respectively functions of u and v only and a is a constant. For the element of length we get

$$ds^2 = du^2 + (U - V)^2\, dv^2$$

and

$$u = \text{const.}, \quad v = \text{const.}$$

are the lines of curvature of this surface. Here

$$E = 1, \quad F = 0, \quad G = (U - V)^2,$$

$$\frac{1}{R_1} = -\frac{1}{2G\sqrt{E}}\,\frac{\partial G}{\partial u} = -\frac{U'}{U - V},$$

$$\frac{1}{R_2} = 0, \quad \rho_1 = \frac{E}{L} = -\frac{\sqrt{1 - U'^2}}{U''},$$

$$\rho_2 = \frac{G}{N} = \frac{(U - V)^2}{\sqrt{1 - U'^2}},$$

where

$$L = -\frac{U''}{\sqrt{1 - U'^2}}, \quad N = \sqrt{1 - U'^2}, \quad M = 0$$

are the 2[nd] fundamental coefficients of Gauss. Hence

$$\frac{1}{h_1^2} = \frac{1}{\rho_1^2} = \frac{U''^2}{1 - U'^2} = f(u),$$

and since $s_1 = u$, we have

$$h_1^2 = f(s_1).$$

The focal surface will be a developable surface. Its linear element is

$$ds_1^2 = [d(u + R_1)]^2 + \frac{R_1^2 U''^2}{1 - U'^2}\cdot du^2,$$

or making

$$u + R_1 = t, \quad \int \frac{U''}{\sqrt{1 - U'^2}}\, du = \text{arc sin } U' = Z,$$

we have

$$ds_1^2 = dt^2 + (t - F(Z))^2\, dZ.$$

To finish this subject, we prove a theorem suggested by a note of T. Caronnet.*

*Th. Caronnet, C. R. 1892, "Sur les centres de courbure géodésique." The following proof is identical with the one given by Caronnet.

We are going

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e the lines of curvature of (S_1).

Again, if

$$\frac{R_1}{h_1} = m,$$

here m is a constant, we obtain a developable surface, and as previously marked, this may take place only when the corresponding lines of curvature e plane curves.

If we suppose

$$\frac{R_1}{h_1} = e^{2q_1} \quad \text{or} \quad \frac{e^{q_1} \pm e^{-q_1}}{2},$$

e shall have for (S_1) a surface of revolution whose total curvature is constant.

§5.

We pass now to the elements of the 2^{nd} order of the focal surfaces (S_1) and S_2). We must find the rotations p', q', p_1', q_1', r_1' of (S_1) and $p'', q'', r'', p_1'', q_1'', r_1''$ (S_2). We know that if a, b, c and the accented letters are the direction sines of the movable axes with respect to the fixed axes, according to the table

	X	Y	Z
x	a	a'	a''
y	b	b'	b''
z	c	c'	c''

X, Y, Z are the fixed and x, y, z the movable axes), then the rotations are pressed by the formulæ

$$p = \Sigma c \frac{db}{du}, \quad q = -\Sigma c \frac{da}{du}, \quad r = \Sigma b \frac{da}{du},$$

$$p_1 = \Sigma c \frac{db}{dv}, \quad q_1 = -\Sigma c \frac{da}{dv}, \quad r_1 = \Sigma b \frac{da}{dv}.$$

e have already remarked that in order to pass from (T) to (T_1) or (T_2) we ust rotate the trihedron (T) around the axes x and y through the angles ϖ_1 nd ϖ_2 respectively.

The following is a table of direction cosines of the movable axes x, y, z

with the fixed axes and with the axes x', y', z', x'', y'', z'' of the trihedrons (T_1) and (T_2):

	X	Y	Z	x'	y'	z'	x''	y''	z''
x	a	a'	a''	1	0	0	$\sin \varpi_2$	0	$-\cos \varpi_2$
y	b	b'	b''	0	$\sin \varpi_1$	$\cos \varpi_1$	0	1	0
z	c	c'	c''	0	$\cos \varpi_1$	$-\sin \varpi_1$	$\cos \varpi_2$	0	$\sin \varpi_2$

Hence we can write

$$\begin{aligned}
&\cos (Xx') = a, \quad \cos (Yx') = a', \quad \cos (Zx') = a'', \\
&\cos (Xy') = b \sin \varpi_1 + c \cos \varpi_1, \\
&\cos (Yy') = b' \sin \varpi_1 + c' \cos \varpi_1, \\
&\cos (Zy') = b'' \sin \varpi_1 + c'' \cos \varpi_1, \\
&\cos (Xz') = b \cos \varpi_1 - c \sin \varpi_1, \\
&\cos (Yz') = b' \cos \varpi_1 - c' \sin \varpi_1, \\
&\cos (Zz') = b'' \cos \varpi_1 - c'' \sin \varpi_1, \\
&\cos (Xx'') = a \sin \varpi_2 + c \cos \varpi_2, \\
&\cos (Yx'') = a' \sin \varpi_2 + c' \cos \varpi_2, \\
&\cos (Zx'') = a'' \sin \varpi_2 + c'' \cos \varpi_2, \\
&\cos (Xy'') = b, \quad \cos (Yy'') = b', \quad \cos (Zy'') = b'', \\
&\cos (Xz'') = - a \cos \varpi_2 + c \sin \varpi_2, \\
&\cos (Yz'') = - a' \cos \varpi_2 + c' \sin \varpi_2, \\
&\cos (Zz'') = - a'' \cos \varpi_2 + c'' \sin \varpi_2.
\end{aligned}$$

We can now write the expressions for the rotations. Namely,

$$\begin{aligned}
p' &= \Sigma c_1 \frac{db_1}{du} = p + \frac{\partial \varpi_1}{\partial u}, \\
p'_1 &= p_1 + \frac{\partial \varpi_1}{\partial v}, \\
q' &= - \Sigma c_1 \frac{da_1}{du} = - (r \cos \varpi_1 + q \sin \varpi_1), \\
q'_1 &= - (r_1 \cos \varpi_1 + q_1 \sin \varpi_1), \\
r' &= \Sigma b_1 \frac{da_1}{du} = r \sin \varpi_1 - q \cos \varpi_1, \\
r'_1 &= r_1 \sin \varpi_1 - q_1 \cos \varpi_1.
\end{aligned}$$

And similarly,

$$p'' = r \cos \varpi_2 + p \sin \varpi_2,$$
$$p_1'' = r_1 \cos \varpi_2 + p_1 \sin \varpi_2,$$
$$q'' = q + \frac{\partial \varpi_2}{\partial u},$$
$$q_1'' = q_1 + \frac{\partial \varpi_2}{\partial v},$$
$$r'' = r \sin \varpi_2 - p \cos \varpi_2,$$
$$r_2'' = r_1 \sin \varpi_2 - p_1 \cos \varpi_2.$$

Introducing in these formulæ the values of p, q, r, p_1, q_1, r_1, $\cos \varpi_1$, $\sin \varpi_1$, $\cos \varpi_2$, $\sin \varpi_2$, we obtain the following table:

$$\begin{aligned}
p' &= \frac{\partial \varpi_1}{\partial u} = \frac{\sqrt{E}}{\tau_1}, & q' &= 0, \\
p_1' &= p_1 + \frac{\partial \varpi_1}{\partial v}, & q_1' &= \frac{h_1 \sqrt{G}}{R_1 \rho_1}, \\
r' &= \frac{\sqrt{E}}{h_1}, & r_1' &= -\frac{h_1 \sqrt{G}}{R_1 R_2}, \\
p'' &= -\frac{h_2 \sqrt{E}}{\rho_2 R_2}, & p_1'' &= 0, \\
q'' &= q + \frac{\partial \varpi_2}{\partial u}, & q_1'' &= \frac{\partial \varpi_2}{\partial v} = \frac{\sqrt{G}}{\tau_2}, \\
r'' &= -\frac{h_2 \sqrt{E}}{R_1 R_2}, & r_1'' &= -\frac{\sqrt{G}}{h_2}.
\end{aligned}$$

Still following Darboux's notations according to which the 2nd fundamental coefficients of Gauss, L, M, N are given by

$$L = \frac{D}{\Delta}, \quad M = \frac{D'}{\Delta}, \quad N = \frac{D''}{\Delta},$$

where the D's are given by the equations

$$\left.\begin{aligned}
\Delta^2 p &= \xi D' - \xi_1 D; & \Delta^2 q &= \eta D' - \eta_1 D, \\
\Delta^2 p_1 &= \xi D'' - \xi_1 D'; & \Delta^2 q_1 &= \eta D'' - \eta_1 D',
\end{aligned}\right\} \tag{16}$$

and

$$\Delta = \eta_1 \xi - \eta \xi_1, \tag{17}$$

we get, by applying successively the above formulæ to the surfaces (S_1) and (S_2),

$$\Delta' = -\eta'\xi_1', \quad \Delta'' = -\eta''\xi_1''. \tag{18}$$

Then for (S_1): $q_1' = 0$, $\eta_1' = 0$; hence

$$D_1' = 0,$$

and for (S_2): $p_1'' = 0$, $\xi'' = 0$; hence

$$D_2' = 0.$$

That is,

$$M_1 = 0, \quad M_2 = 0,$$

and consequently the lines u and v on the surfaces (S_1) and (S_2) are conjugate. This is a well-known property of the focal surfaces.

Now

$$L_1 = -\frac{D_1}{\Delta_1} = -\frac{R_1}{h_1}\sqrt{E}.\frac{\partial \varpi_1}{\partial u},$$

or

$$L_1 = -\frac{R_1 E}{h_1 \tau_1}$$

and

$$N_1 = \frac{D_2}{\Delta_1} = -\frac{\partial R_1}{\partial v}\cdot\frac{h_1\sqrt{G}}{R_1\rho_1}.$$

Similarly

$$L_2 = -\frac{\sqrt{E}.h_2}{R_2\rho_2}\cdot\frac{\partial R_2}{\partial u}, \quad N_2 = -\frac{R_2 G}{h_2\tau_2}. \tag{19}$$

The measures of total curvature of (S_1) and (S_2) will be

for (S_1)

$$\frac{1}{\rho_1'\rho_2'} = \frac{1}{\frac{\partial R_1}{\partial v}}\cdot\frac{h_1^2\sqrt{G}}{\tau_1 R_1^2\rho_1};$$

for (S_2),

$$\frac{1}{\rho_1''\rho_2''} = \frac{1}{\frac{\partial R_2}{\partial u}}\cdot\frac{h_2^2\sqrt{E}}{\tau_2 . R_2^2\rho_2}.$$

In order that any of these expressions may become zero, i. e. in order that any of ρ_1', ρ_2', ρ_1'', ρ_2'' may become infinitely large and the focal surface may become a developable surface, we must have any one of the following equations:

$$\tau_1 = \infty, \quad \tau_2 = \infty, \quad R_1 = \infty, \quad R_2 = \infty, \quad \rho_1 = \infty, \quad \rho_2 = \infty.$$

All of these equations show that the lines of curvature of (Σ) in one system or in both systems must be plane curves. And conversely, if (S_1) or (S_2) be developable surfaces, the corresponding lines of curvature must be plane curves.

The expressions for the rotations p_1' and q'' give us a means of finding the derivatives

$$\frac{\partial \varpi_1}{\partial v}, \quad \frac{\partial \varpi_2}{\partial u}.$$

From
$$\Delta_1^2 p_1' = \xi' D_1'', \quad \Delta_{11}^2 q'' = -\eta_1'' D_{11}$$

we have
$$p_1' = \frac{D_1''\xi'}{\Delta_1^2} = -\frac{h_1^2}{R_1^2\rho_1}\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) \cdot \sqrt{\frac{G}{E}}$$

and
$$\frac{\partial \varpi_1}{\partial v} = -\left[\frac{1}{h_2} + \frac{h_1^2}{\sqrt{E}\,.\,R_1^2\rho_1}\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right)\right] . \sqrt{G},$$

or remarking that the derivatives

$$\frac{\partial}{\sqrt{G}\,.\,\partial v} \text{ and } \frac{\partial}{\sqrt{E}\,.\,\partial u}$$

represent the derivatives with respect to the displacements along the lines of curvature ds_2 and ds_1, we can write

$$\frac{\partial \varpi_1}{\partial s_2} = -\left[\frac{1}{h_2} + \frac{h_1^2}{\rho_1 R_1^2}\left(\frac{\partial R_1}{\partial s_1} + 1\right)\right].$$

In the same way we have

$$q'' = -\left[\frac{\partial R_2}{\partial v} + \sqrt{G}\right] . \frac{h_2}{R_2^2\rho_2} . \sqrt{\frac{E}{G}}$$

and
$$\frac{\partial \varpi_2}{\partial s_1} = -\left[\frac{1}{h_1} + \frac{h_2^2}{\rho_2 R_2^2}\left(\frac{\partial R_2}{\partial s_2} + 1\right)\right].$$

§6.

The equations of the asymptotic lines are:

for (S_1),
$$\frac{R_1^2}{h_1^2} . \sqrt{E} . \frac{\partial \varpi_1}{\partial u} . du^2 + \frac{1}{\rho_1} . \frac{\partial R_1}{\partial v} . \sqrt{G} . dv^2 = 0;$$

for (S_2),
$$\frac{R_2^2}{h_2^2} . \sqrt{G} . \frac{\partial \varpi_2}{\partial v} . dv^2 + \frac{1}{\rho_2} . \frac{\partial R_2}{\partial u} . \sqrt{E} . du^2 = 0.$$

By inspection we find from these equations that in order that the lines of curvature of (Σ) correspond to the asymptotic lines of (S_1) and (S_2), we must have simultaneously

$$\frac{\partial \varpi_1}{\partial u}=0 \quad , \quad \frac{\partial \varpi_2}{\partial v}=0 \quad , \tag{20}$$

or

$$\varpi_1=f(v), \quad \varpi_2=f_1(u),$$

i. e. the angles between the osculating planes of the u and the v lines and the tangent planes to (Σ) must be constant. The converse is also correct. But the equations (20) give

$$\tau_1=\infty, \quad \tau_2=\infty,$$

i. e. that the lines of curvature of (Σ) in both systems must be plane.

We know all the surfaces having lines of curvature plane in both systems. They are the envelopes of the plane

$$\begin{aligned} \alpha x-\beta y &+(\lambda\sqrt{1-x^2}-\sqrt{\lambda^2-1}\,.\sqrt{1+\beta^2})\,z \\ &=f(\alpha)-f(\beta), \end{aligned}$$

where α and β are parameters of the lines of curvature. Hence we know all the surfaces for which the lines of curvature correspond to the asymptotic lines of the focal surfaces (S_1) and (S_2).

The equations of the asymptotic lines of (Σ) are

$$\frac{E}{\rho_1}\,du^2+\frac{G}{\rho_2}\,dv^2=0.$$

In order that the asymptotic lines of (Σ) may correspond to those of (S_1) we must have

$$\left.\begin{aligned} \frac{E}{\rho_1}&=m\,.\,\frac{R_1^2}{h_1^2}\,.\sqrt{E}\,.\,\frac{\sqrt{E}}{\tau_1}, \\ \frac{G}{\rho_2}&=m\,.\,\frac{1}{\rho_1}\,.\,\frac{\partial R_1}{\partial v}\,.\sqrt{G}. \end{aligned}\right\} \tag{21}$$

Now we have seen that for (S_1)

$$\frac{1}{\rho_1'\rho_2'}=\frac{1}{\frac{\partial R_1}{\partial v}}\cdot\frac{h_1^2\sqrt{G}}{\tau_1\,.\,R_1^2\rho_1},$$

and from the equations (21),

$$\frac{1}{\rho_1\rho_2}=\frac{1}{\rho_1^3}\cdot\frac{\partial R_1}{\partial v}\cdot\frac{h_1^2}{R_1^2}\cdot\tau_1\cdot\frac{1}{\sqrt{G}},$$

and finally,

$$\rho_1' \rho_2' \rho_1 \cdot \rho_2 = \left(\frac{R_1 \rho_1}{h_1}\right)^4.$$

Calling

$$V \equiv \frac{\pi}{2} - \varpi_1,$$

we get

$$h_1' h_2' h_1 h_2 = \left(\frac{R_1}{\sin V}\right)^4.$$

This is a particular case of a theorem given by E. Cosserat* and A. Demoulin.† It follows at once that if we have the above relation between the radii of principal curvatures of (Σ) and (S_1), their asymptotic lines correspond. For then

$$\rho_1' \rho_2' = \frac{1}{\rho_1 \rho_2} \cdot \frac{R_1^4 \rho_1^4}{h_1^4},$$

and since for (S_1) we have

$$\frac{1}{\rho_1' \rho_2'} = \frac{1}{\frac{\partial R_1}{\partial v}} \cdot \frac{h_1^2 \sqrt{G}}{\tau_1 \cdot R_1^2 \rho_1},$$

the above equation becomes

$$\frac{1}{\frac{1}{\frac{\partial R_1}{\partial v}} \cdot \frac{h_1^2 \sqrt{G}}{\tau \cdot R_1^2 \rho_1}} = \frac{1}{\rho_1 \rho_2} \cdot \frac{R_1^4 \rho_1^4}{h_1^4},$$

and hence

$$\frac{E}{\rho_1} : \frac{G}{\rho_2} = \frac{R_1^2}{h_1^2} \frac{E}{\tau_1} : \frac{1}{h_1} \frac{\partial R_1}{\partial v} \sqrt{G},$$

which shows that the two equations of the asymptotic lines for (Σ) and (S_1) are identical.

So that we can say: if the product of four radii of principal curvatures of (Σ) and (S_1) is equal to the fourth power of the fourth proportional to R_1, ρ_1 and h_1, the asymptotic lines on both surfaces correspond. And if we desire to have the asymptotic lines of (S_2) correspond to those of (S_2), we must have

$$\frac{R_1^2}{h_1^2} \cdot \sqrt{E} \cdot \frac{\sqrt{E}}{\tau_1} = m \cdot \frac{1}{\rho_2} \cdot \frac{\partial R_2}{\partial u} \cdot \sqrt{E},$$

$$\frac{1}{\rho_1} \cdot \frac{\partial R_1}{\partial v} \cdot \sqrt{G} = m \cdot \frac{R_2^2}{h_2^2} \cdot \sqrt{G} \cdot \frac{\sqrt{G}}{\tau_2}.$$

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* E. Cosserat, C. R., vol. 118, 1894, "Sur des congruences rectilignes et sur le problème de Ribaucour."

† A. Demoulin, C. R., 1894, "Sur une propriété métrique commune à trois classes particulières de congruences rectilignes."

Now we had

$$\frac{1}{\rho_1'\rho_2'} = \frac{1}{\dfrac{\partial R_1}{\partial v}} \cdot \frac{h_1^2\sqrt{G}}{\tau_1 . R_1^2\rho_1},$$

$$\frac{1}{\rho_2''\rho_1''} = \frac{1}{\dfrac{\partial R_2}{\partial u}} \cdot \frac{h_2^2\sqrt{E}}{\tau_2 R_2^2\rho_2},$$

or

$$\rho_1'\rho_2'\rho_1''\rho_2'' = \frac{\tau_1\tau_2\rho_1\rho_2 R_1^2 R_2^2}{h_1^2 h_2^2\sqrt{EG}} \cdot \frac{\partial R_1}{\partial v} \cdot \frac{\partial R_2}{\partial u}.$$

From the first two equations

$$\frac{\partial R_1}{\partial u} \cdot \frac{\partial R_2}{\partial v} \cdot \frac{\tau_1\tau_2}{\sqrt{EG}} = \frac{R_1^2 R_2^2\rho_1\rho_2}{h_1^2 h_2^2},$$

hence

$$\rho_1'\rho_2'\rho_1''\rho_2'' = \frac{R_1^4 R_2^4\rho_1^2\rho_2^2}{h_1^4 h_2^4},$$

or if we want to introduce the angles ϖ_1 and ϖ_2 we have

$$\frac{R_1}{h_2} = \frac{1}{\sin\varpi_2}, \quad \frac{R_2}{h_1} = \frac{1}{\sin\varpi_1}$$

and

$$\rho_1'\rho_2'\rho_1''\rho_2'' = \left(\frac{\rho_1\rho_2}{\sin^2\varpi_1\sin^2\varpi_2}\right)^2,$$

or again,

$$\rho_1'\rho_2'\rho_1''\rho_2'' = \left(\frac{2R_1 . 2R_2}{\sin 2\varpi_1 \sin 2\varpi_2}\right)^2.$$

In a way identical to that on page 126, we can prove that this relation is also sufficient for or characteristic of the correspondence of the asymptotic lines on (S_1) and (S_2).

From the relations

$$\rho_1'\rho_2'\rho_1\rho_2 = \left(\frac{R_1\rho_1}{h_1}\right)^4$$

and

$$\rho_1'\rho_2'\rho_1''\rho_2'' = \frac{R_1^4 R_2^4\rho_1^2\rho_2^2}{h_1^4 h_2^4},$$

we conclude that if there exists a correspondence of the asymptotic lines on (Σ) and (S_1) or (S_1) and (S_2), then at corresponding points both surfaces are either convex or of opposite curvature. For instance, if $\rho_1' > 0$, $\rho_2' > 0$, then $\rho_1^{(\kappa)}$ and

$\rho_2^{(\varkappa)}$ must be either both $>$ or both < 0, $(\varkappa = 0, 2)$. If on the contrary $\rho_1' > 0$, $\rho_2' < 0$, then $\rho_1^{(\varkappa)} \gtrless 0$ and $\rho_2^{(\varkappa)} \lesseqgtr 0$.

Again for such surfaces, if (Σ) has its total curvature constant, (S_1) can be of the same kind only if

$$\frac{R_1 \rho_1}{h_1} = \text{const.}$$

Since the equations of the asymptotic lines and the equations connecting any two conjugate directions du and δu, dv and δv, contain the same coefficients, we may say that when the above relations, connecting the four radii of principal curvature, exist, then the conjugate directions of (Σ) and (S_1) or of (S_1) and (S_2) correspond.

§7.

The equations of the lines of curvature we write in two different ways, viz. For (S_1) (calling ρ_1^0, ρ_2^0 the radii of principal curvature of (S_1) and (S_2),

$$\left.\begin{aligned} \left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) du + \frac{\partial R_1}{\partial v} dv + \rho_1^0 \frac{h_1 \sqrt{G}}{R_1 \rho_1} dv &= 0, \\ \frac{R_1}{h_1} \sqrt{E}\, du + \rho_1^0 \left(\frac{\partial \varpi_1}{\partial u} du + p_1 dv + \frac{\partial \varpi_1}{\partial v} dv\right) &= 0. \end{aligned}\right\} \tag{22}$$

For (S_2),

$$\left.\begin{aligned} \left(\frac{\partial R_2}{\partial v} + \sqrt{G}\right) dv + \frac{\partial R_2}{\partial u} du + \rho_2^0 \frac{h_2 \sqrt{E}}{R_2 \rho_2} du &= 0, \\ \frac{R_2}{h_2} \sqrt{G} . dv + \rho_2^0 \left(\frac{\partial \varpi_2}{\partial v} . dv + q du + \frac{\partial \varpi_2}{\partial u} du\right) &= 0. \end{aligned}\right\} \tag{23}$$

Eliminating from these four equations ρ_1^0 and ρ_2^0, we obtain the differential equations of the lines of curvature of (S_1) and (S_2).

For (S_1),

$$\begin{aligned} &\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) . \frac{\rho_1}{\tau_1} . \sqrt{E}\, du^2 + \left(\frac{\partial R_1}{\partial v} . \frac{\rho_1}{\tau_1} . \sqrt{E} - \sqrt{EG} \right. \\ &\qquad \left. - \sqrt{\frac{G}{E}} . \frac{h_1^2}{R_1^2} \left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right)^2\right) du\, dv \\ &\qquad - \frac{h_1^2}{R_1^2} \left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) . \sqrt{\frac{G}{E}} . \frac{\partial R_1}{\partial v} dv^2 = 0, \end{aligned}$$

and for (S_2),

$$\left(\frac{\partial R_2}{\partial v}+\sqrt{G}\right)\frac{\rho_2}{\tau_2}\sqrt{G}\,.\,dv^2+\left(\frac{\partial R_2}{\partial u}\cdot\frac{\rho_2}{\tau_2}\cdot\sqrt{G}-\sqrt{EG}\right.$$
$$-\sqrt{\frac{E}{G}}\cdot\frac{h_2^2}{R_2^2}\cdot\left.\left(\frac{\partial R_2}{\partial v}+\sqrt{G}\right)^2\right)du\,dv$$
$$-\frac{h_2^2}{R_2^2}\left(\frac{\partial R_2}{\partial v}+\sqrt{G}\right)\cdot\sqrt{\frac{E}{G}}\cdot\frac{\partial R_2}{\partial u}du^2=0.$$

From these equations we can see that if we want u and v to remain parameters of the lines of curvature on both focal surfaces (S_1) and (S_2), we must have simultaneously

$$\frac{\partial R_1}{\partial u}+\sqrt{E}=0,\quad \frac{\partial R_2}{\partial v}+\sqrt{G}=0.$$

The first equation gives

$$\frac{\partial R_1}{\partial u}=-\sqrt{E},$$

and since
$$R_1=-\frac{\sqrt{G}}{r_1},$$

$$\frac{\partial R_1}{\partial u}=-\frac{1}{r_1}\cdot\frac{\partial\sqrt{G}}{\partial u}+\frac{\sqrt{G}}{r_1^2}\cdot\frac{\partial r_1}{\partial u}=-\sqrt{E}.$$

Again,
$$r_1=\frac{1}{\sqrt{E}}\cdot\frac{\partial\sqrt{G}}{\partial u},\quad \sqrt{E}=\frac{1}{r_1}\cdot\frac{\partial\sqrt{G}}{\partial u},$$

hence
$$\frac{\sqrt{G}}{r_1^2}\cdot\frac{\partial r_1}{\partial u}=0,$$

i. e.
$$\frac{\partial r_1}{\partial u}=0.$$

The second equation gives

$$\frac{\partial R_2}{\partial v}=-\sqrt{G}=\frac{1}{r}\cdot\frac{\partial\sqrt{E}}{\partial v}-\frac{\sqrt{E}}{r^2}\cdot\frac{\partial r}{\partial v},$$

and since
$$r=-\frac{1}{\sqrt{G}}\cdot\frac{\partial\sqrt{E}}{\partial v};\quad \sqrt{G}=-\frac{1}{r}\cdot\frac{\partial\sqrt{E}}{\partial v},$$

we have
$$-\frac{\sqrt{E}}{r^2}\cdot\frac{\partial r}{\partial v}=0,$$

i. e.
$$\frac{\partial r}{\partial v}=0.$$

Hence
$$\frac{\partial r}{\partial v} - \frac{\partial r}{\partial u} = - qp_1 = 0\,,$$

which requires either $q = 0$ or $p_1 = 0$.

If $q = 0$,
$$\frac{\partial q}{\partial v} = \quad rp_1 = 0 \text{ and } p_1 = 0\,;$$

if $p_1 = 0$,
$$\frac{\partial p_1}{\partial u} = - qr_1 = 0 \text{ and } q = 0\,.$$

We excluded from consideration the cases where r or r_1 are equal to zero, for then the surface (Σ) would belong to the class of surfaces enumerated by Darboux (l. c. v. III, p. 122) and the focal surface would degenerate into a line, so that we have
$$p_1 = 0\ ,\quad q = 0\ ,$$
i. e.
$$\rho_1 = \infty\,,\quad \rho_2 = \infty\,,$$

and we get a trivial solution of the plane. This theorem has been proved by Professor Craig in a somewhat different way.

The question, can the lines of curvature of (Σ) correspond to those of say (S_1)? has been treated by C. Guichard* in connection with the surfaces of constant total curvature.

Let us eliminate from the equations (22) and (23) $\frac{dv}{du}$; we obtain equations determining radii of principal curvature of (S_1) and (S_2).

For (S_1),
$$\frac{\partial \varpi_1}{\partial u}\ \frac{h_1}{R_1}\ \frac{\sqrt{G}}{\rho_1}\,\rho^2 - \rho\ \left[\left(p_1 + \frac{\partial \varpi_1}{\partial v}\right)\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) - \frac{\sqrt{EG}}{h_1} - \frac{\partial R_1}{\partial v}\cdot\frac{\partial \varpi_1}{\partial u}\right] + \frac{R_1 . \sqrt{E}}{h_1}\cdot\frac{\partial R_1}{\partial v} = 0\,.$$

For (S_2),
$$\frac{\partial \varpi_2}{\partial v}\ \frac{h_2\sqrt{E}}{R_2\,\rho_2}\,\rho^2 - \rho\left[\left(q + \frac{\partial \varpi_2}{\partial u}\right)\left(\frac{\partial R_2}{\partial v} + \sqrt{G}\right) - \frac{\sqrt{EG}}{h_2} - \frac{\partial R_2}{\partial u}\cdot\frac{\partial \varpi_2}{\partial v}\right] + \frac{R_2\sqrt{G}}{h_2}\cdot\frac{\partial R_2}{\partial u} = 0\,.$$

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* C. Guichard, "Recherches sur les surfaces à courbure total constante," etc. (Ann. de l'École Normale, 1890).

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Calling K_1 and K_2 the respective total curvatures of (S_1) and (S_2) and H_1 and H_2 their mean curvatures, we have

$$H_1 = \frac{h_1\left[\left(p_1+\frac{\partial \varpi_1}{\partial v}\right)\left(\frac{\partial R_1}{\partial u}+\sqrt{E}\right)-\frac{\sqrt{EG}}{\rho_1}-\frac{\partial R_1}{\partial v}\cdot\frac{\partial \varpi_1}{\partial u}\right]}{R_1\sqrt{E}\,.\,\frac{\partial R_1}{\partial v}},$$

$$K_1 = \frac{h_1^2\sqrt{G}\,.\,\frac{\partial \varpi_1}{\partial u}}{R_1^2\rho_1\frac{\partial R_1}{\partial v}\,.\,\sqrt{E}},$$

$$H_2 = \frac{h_2\left[\left(q+\frac{\partial \varpi_2}{\partial u}\right)\left(\frac{\partial R_2}{\partial v}+\sqrt{G}\right)-\frac{\sqrt{EG}}{\rho_2}-\frac{\partial R_2}{\partial u}\cdot\frac{\partial \varpi_2}{\partial v}\right]}{R_2\sqrt{G}\,.\,\frac{\partial R_1}{\partial v}},$$

$$K_2 = \frac{h_2^2\sqrt{E}\,.\,\frac{\partial \varpi_2}{\partial v}}{R_2^2\,.\,\rho_2\,.\,\frac{\partial R_2}{\partial u}\sqrt{G}}.$$

In case the lines of curvature of (Σ) correspond to those of (S_1), we have

$$\frac{\partial R_1}{\partial u}+\sqrt{E}=0,$$

$$H_1 = -\frac{\sqrt{EG}+\frac{\partial R_1}{\partial v}\cdot\frac{\partial \varpi_1}{\partial u}}{R_1\,.\sqrt{E}\,.\,\frac{\partial R_1}{\partial v}}; \quad K_1 = \frac{h_1^2\sqrt{G}\,.\,\frac{\partial \varpi_1}{\partial u}}{R_1^2\rho_1\frac{\partial R_1}{\partial v}\,.\sqrt{E}}.$$

In this case we can write very simple expressions for the radii of principal curvatures. Since u and v are parameters of lines of curvature on (S_1), we have

$$\rho_1' = \frac{E_1}{L_1} = -\frac{R_1^2E}{h_1^2}\cdot\frac{h_1\tau_1}{ER_1} = -\frac{R_1\tau_1}{h_1},$$

$$\rho_2' = \frac{G_1}{N_1} = -\frac{\left(\frac{\partial R_1}{\partial v}\right)^2.\,R_1\rho_1}{h_1\,.\sqrt{G}\,.\,\frac{\partial R_1}{\partial v}} = -\frac{R_1\rho_1}{h_1\sqrt{G}}\cdot\frac{\partial R_1}{\partial v}.$$

Thus we see that ρ_1' is the fourth proportional to R_1, τ_1, h_1. This property furthermore is characteristic of the surfaces. For suppose that one of the radii of curvature of the focal surface (S_1), of the congruence of tangents to the line of

curvature $v=$ const. of the surface (Σ), is the fourth proportional to R_1, τ_1, h_1. Then for the second radius of curvature we shall have

$$\rho_2' = -\frac{R_1 \rho_1}{h_1 \sqrt{G}} \cdot \frac{\partial R_1}{\partial v},$$

and this requires

$$\left(p_1 + \frac{\partial \varpi_1}{\partial v}\right)\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) = 0,$$

or

$$\frac{h_1^2 \sqrt{G}}{\sqrt{E} \,.\, R_1^2 \rho_1}\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right)^2 = 0,$$

or

$$\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) = 0. \qquad \text{Q. E. D.}$$

Therefore the necessary and sufficient condition that the lines of curvature on (Σ) and (S_1) correspond is that

$$\rho_1' = \frac{R_1 \tau_1}{h_1}.$$

Consider now two surfaces (S_1) and (S_2) such that the lines of curvature of one correspond to those of the other. We must have

$$\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) \cdot \frac{\rho_1}{\tau_1} \,.\, \sqrt{E} = -m \,.\, \frac{h_2^2}{R_2^2} \cdot \left(\frac{\partial R_2}{\partial v} + \sqrt{G}\right) \cdot \sqrt{\frac{E}{G}} \cdot \frac{\partial R_2}{\partial u},$$

$$-\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) \cdot \frac{h_1^2}{R_1^2} \cdot \sqrt{\frac{G}{E}} \cdot \frac{\partial R_1}{\partial v} = m \,.\left(\frac{\partial R_2}{\partial v} + \sqrt{G}\right) \cdot \frac{\rho_2}{\tau_2} \,.\, \sqrt{G},$$

$$\frac{\partial R_1}{\partial v} \cdot \frac{\rho_1}{\tau_1} \,.\, \sqrt{E} - \sqrt{\frac{G}{E}} \cdot \frac{h_1^2}{R_1^2}\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right)^2 = m\left[\frac{\partial R_2}{\partial u} \cdot \frac{\rho_2}{\tau_2}\sqrt{G} - \sqrt{\frac{E}{G}} \,\cdot\right.$$

$$\left. \cdot \frac{h_2^2}{R_2^2}\left(\frac{\partial R_2}{\partial v} + \sqrt{G}\right)^2\right].$$

The first two equations give

$$\frac{\rho_1 \sqrt{E} \,.\, R_1^2}{\tau_1 \,.\, h_1^2 \dfrac{\partial R_1}{\partial v}} = \frac{h_2^2 \,.\, \dfrac{\partial R_2}{\partial u} \,.\, \tau_2}{\rho_2 \,.\, \sqrt{G} \,.\, R_2^2},$$

or

$$\frac{\tau_1 \tau_2}{\sqrt{EG}} \cdot \frac{\partial R_1}{\partial v} \cdot \frac{\partial R_2}{\partial u} = \frac{\rho_1 \rho_2 R_1^2 R_2^2}{h_1^2 h_2^2}$$

and

$$h_1' h_2' h_1'' h_2'' = \frac{R_1^4 \,.\, R_2^4 \rho_1^2 \rho_2^2}{h_1^4 h_2^4},$$

which is identical with the necessary and sufficient condition that (S_1) and (S_2) have their asymptotic lines corresponding. In case of corresponding lines of

curvature this is a necessary condition, but it is not sufficient, as the third equation written above has to be satisfied. We can therefore state: if the lines of curvature on (S_1) and (S_2) correspond, the asymptotic lines correspond also, but not conversely.

§8.

For the spherical representation of an element of a curve on the focal surfaces (S_1) and (S_2) we easily obtain from the general formula:

$$d\sigma_\kappa^2 = (p^{(\kappa)}\,du + p_1^{(\kappa)}\,dv)^2 + (q^{(\kappa)}\,du + q_1^{(\kappa)}\,dv)^2.$$

For (S_1),

$$d\sigma_1^2 = \left(d\varpi_1 + \frac{\sqrt{G}\,dv}{\rho_2}\right)^2 + \frac{h_1^2\rho_2^2}{R_1^2\rho_1^2}\left(\frac{\sqrt{G}\,dv}{\rho_2}\right)^2.$$

For (S_2),

$$d\sigma_2^2 = \left(d\varpi_2 + \frac{\sqrt{E}\,.\,du}{\rho_1}\right)^2 + \frac{h_2^2\rho_1^2}{R_2^2\rho_2^2}\,.\left(\frac{\sqrt{E}\,du}{\rho_1}\right)^2,$$

or

$$d\sigma_1^2 = \left(d\varpi_1 + \frac{ds_2}{\rho_2}\right)^2 + \frac{h_1^2}{R_1^2\rho_1^2}\,.\,ds_2^2,$$

$$d\sigma_2^2 = \left(d\varpi_2 + \frac{ds_1}{\rho_1}\right)^2 + \frac{h_2^2}{R_2^2\rho_2^2}\,.\,ds_1^2.$$

These two formulæ evidently do not depend on the choice of the coordinate lines.

§9.

Let us now consider the congruence of straight lines tangent to any given family of lines on the surface (Σ).

Suppose that the surface is given by $x = f(u, v)$, $y = \phi(u, v)$, $z = \psi(u, v)$. The lines on the surface are given by their differential equation, from which we can obtain the values of $\frac{dv}{du} = \lambda_\kappa$.

Then the coordinates of the focal surface are given by

$$x_1 = x + \alpha_1 R_1;\quad y_1 = y + \beta_1 R_1;\quad z_1 = z + \gamma_1 R_1,$$

where α_1, β_1, γ_1 are the direction cosines of the tangents to the given curve and represented by

$$\alpha_1 = \frac{\frac{\partial x}{\partial u} + \lambda_\kappa\,.\,\frac{\partial x}{\partial v}}{\sqrt{E + 2F\lambda_\kappa + G\lambda_\kappa^2}}\ \text{etc.}$$

R_1 is the focal distance and given according to Kummer, by

$$R_1 = -\frac{gE' - (f+f')F' + eG'}{E'G' - F'^2},$$

where

$$e = \Sigma\frac{\partial x}{\partial u}\cdot\frac{\partial \alpha_1}{\partial u}, \quad f = \Sigma\frac{\partial x}{\partial v}\cdot\frac{\partial x_1}{\partial u}, \quad f' = \Sigma\frac{\partial x}{\partial u}\cdot\frac{\partial x_1}{\partial v}, \quad g = \Sigma\frac{\partial x}{\partial v}\cdot\frac{\partial \alpha_1}{\partial v},$$

$$E' = \Sigma\left(\frac{\partial x_1}{\partial u}\right)^2, \quad F' = \Sigma\frac{\partial x_1}{\partial u}\cdot\frac{\partial x_1}{\partial v}, \quad G' = \Sigma\left(\frac{\partial x_1}{\partial v}\right)^2.$$

The fundamental coefficients of the focal surface $\overline{E}, \overline{F}, \overline{G}$ we obtain according to the formulæ:

$$\overline{E} = \Sigma\left(\frac{\partial x_1}{\partial u}\right)^2 = \Sigma\left(\frac{\partial x}{\partial u}\right)^2 + R_1^2\Sigma\left(\frac{\partial x_1}{\partial u}\right)^2 + \left(\frac{\partial R_1}{\partial u}\right)^2 + 2R_1\Sigma\frac{\partial x}{\partial u}\cdot\frac{\partial x_1}{\partial u} + 2\frac{\partial R_1}{\partial u}\cdot\Sigma\frac{\partial x}{\partial u}\cdot\alpha_1, \text{ etc.},$$

or

$$\overline{E} = E + R_1^2 E' + \left(\frac{\partial R_1}{\partial u}\right)^2 + 2R_1 e + 2\frac{\partial R_1}{\partial u}\Sigma\alpha_1\cdot\frac{\partial x}{\partial u},$$

$$\overline{F} = F + R_1^2 F' + \frac{\partial R_1}{\partial u}\cdot\frac{\partial R_1}{\partial v} + R_1(f+f') + \frac{\partial R_1}{\partial v}\Sigma\alpha_1\cdot\frac{\partial x}{\partial u} + \frac{\partial R_1}{\partial u}\Sigma\alpha_1\cdot\frac{\partial x}{\partial v},$$

$$\overline{G} = G + R_1^2 G' + \left(\frac{\partial R_1}{\partial v}\right)^2 + 2R_1 g + 2\cdot\frac{\partial R_1}{\partial v}\Sigma\alpha_1\cdot\frac{\partial x}{\partial v}.$$

The linear element ds_1 is therefore

$$ds_1^2 = ds^2 + R_1^2 d\sigma_1^2 + dR_1^2 + 2R_1[e du^2 + (f+f')du dv + g' dv^2] + 2dR_1\{\cos V.\sqrt{E}.du + \cos V_1.\sqrt{G}.dv\},$$

where $$\cos V = \frac{1}{\sqrt{E}}\cdot\Sigma\alpha_1\cdot\frac{\partial x}{\partial u}, \quad \cos V_1 = \frac{1}{\sqrt{G}}\cdot\Sigma\alpha_1\cdot\frac{\partial x}{\partial v}.$$

V and V_1 are the angles between the direction R_1 and the lines $v = \text{const.}$ and $u = \text{const.}$ respectively.

It remains to express

$$d\sigma_1^2 = E' du^2 + 2F' du dv + G' dv^2.$$

For this purpose imagine a sphere of radius 1 and through the center of it draw the lines parallel to the direction R_1; they will describe a certain line on the sphere, the coordinates of which are $\alpha_1, \beta_1, \gamma_1$ and $d\sigma_1^2$ is the element of arc of

what is called the spherical indicatrix of the given line. Its value can be written up from the fact that

$$\frac{d\sigma_1}{ds_\kappa} = \frac{1}{h_\kappa},$$

where h_κ is the radius of the first curvature of the given curve and ds_κ is the element of arc of this curve given by

$$ds_\kappa^2 = E + 2F\lambda_\kappa + G\lambda_\kappa^2.$$

Hence

$$d\sigma_1^2 = \frac{ds_\kappa^2}{h_\kappa^2}.$$

The expression

$$edu^2 + (f + f')\, du\, dv + gdv^2$$

is identically zero. For

$$e = \frac{1}{ds_\kappa}\Big[mdu + m'dv + E\frac{\partial}{\partial u}(du) + F\frac{\partial}{\partial u}(dv) - \frac{1}{ds_\kappa}(E\,du + F\,dv)\frac{\partial}{\partial u}(ds_\kappa)\Big],$$

$$f + f' = \frac{1}{ds_\kappa}\Big[m'du + m''dv + E\,.\frac{\partial}{\partial v}(du) + F\,.\frac{\partial}{\partial v}(dv) - \frac{1}{ds_\kappa}(Edu + Fdv)\,.\frac{\partial}{\partial v}(ds_\kappa) + ndu + n'dv + F\,.\frac{\partial}{\partial u}(du) + G\frac{\partial}{\partial v}(dv) - \frac{1}{ds_\kappa}(Fdu + Gdv)\frac{\partial}{\partial u}(ds_\kappa)\Big],$$

$$g = \frac{1}{ds_\kappa}\Big[n'du + n''dv + F\frac{\partial}{\partial v}(du) + G\,.\frac{\partial}{\partial v}(dv) - \frac{1}{ds_\kappa}(Fdu + Gdv)\frac{\partial}{\partial v}(ds_\kappa)\Big];$$

and hence

$$edu^2 + (f + f')\, du\, dv + gdv^2 = \frac{1}{ds_\kappa}\,[du\,(mdu^2 + 2m'\,du\,dv + m''dv^2 + Ed^2u + Fd^2v) + dv\,(ndu^2 + 2n'du\,dv + n''dv^2 + Fd^2u + Gd^2v] - d^2s_\kappa \equiv 0,$$

where

$$m = \Sigma\frac{\partial x}{\partial u}\cdot\frac{\partial^2 x}{\partial u^2}, \quad m' = \Sigma\frac{\partial x}{\partial u}\cdot\frac{\partial^2 x}{\partial u\partial v}, \quad m'' = \Sigma\frac{\partial x}{\partial u}\cdot\frac{\partial^2 x}{\partial v^2},$$

$$n = \Sigma\frac{\partial x}{\partial v}\cdot\frac{\partial^2 x}{\partial u^2}, \quad n' = \Sigma\frac{\partial x}{\partial v}\cdot\frac{\partial^2 x}{\partial u\partial v}, \quad n'' = \Sigma\frac{\partial^2 x}{\partial v^2}\cdot\frac{\partial x}{\partial v}.$$

Again,
$$\cos V = \frac{Edu + Fdv}{\sqrt{E}\,.\,ds_\kappa},$$
$$\cos V_1 = \frac{Fdu + Gdv}{\sqrt{G}\,.\,ds_\kappa}.$$

Hence the linear element of the focal surface becomes
$$ds_1'^2 = ds_\kappa^2 + R_1^2\,.\,\frac{ds_\kappa^2}{h_\kappa^2} + dR_1^2 + 2dR_1\,.\,ds_\kappa.$$

From this expression we see that for any coordinates u and v
$$\overline{E} = \left(1 + \frac{R_1^2}{h_\kappa^2}\right)\left(\frac{\partial s_\kappa}{\partial u}\right)^2 + \left(\frac{\partial R_1}{\partial u}\right)^2 + 2\,.\,\frac{\partial R_1}{\partial u}\,.\,\frac{\partial s_\kappa}{\partial u},$$
$$\overline{F} = \left(1 + \frac{R_1^2}{h_\kappa^2}\right)\frac{\partial s_\kappa}{\partial u}\,.\,\frac{\partial s_\kappa}{\partial v} + \frac{\partial R_1}{\partial u}\,.\,\frac{\partial R_1}{\partial v} + \frac{\partial R_1}{\partial v}\,.\,\frac{\partial s_\kappa}{\partial u} + \frac{\partial R_1}{\partial u}\,.\,\frac{\partial s_\kappa}{\partial v},$$
$$\overline{G} = \left(1 + \frac{R_1^2}{h_\kappa^2}\right)\left(\frac{\partial s_\kappa}{\partial v}\right)^2 + \left(\frac{\partial R_1}{\partial v}\right) + 2\,.\,\frac{\partial R_1}{\partial v}\,.\,\frac{\partial s_\kappa}{\partial v}.$$

If we write the above expression for ds_1^2 as follows:
$$ds_1'^2 = (ds_\kappa + dR_1)^2 + \frac{R_1^2}{h_\kappa^2}\,.\,ds_\kappa^2 = dq^2 + \frac{R_1^2}{h_\kappa^2}\,ds_\kappa^2,$$
where
$$q = s_\kappa + R_1,$$
we see that
$$s_\kappa = \text{const.}$$
represents a system of geodesics on the focal surface, and
$$s_\kappa + R_1 = \text{const.}$$
are their orthogonal trajectories.

Suppose the surface (Σ) is referred to a system of geodesic lines and their orthogonal trajectories; then $E = 1$, $F = 0$; consider the congruence of lines tangent to the geodesics $v = \text{const.}$; $s_\kappa = u = \text{const.}$ will be geodesics on the focal surface, and since
$$\frac{1}{R_1} = -\frac{\partial \lg \sqrt{G}}{\partial u},$$
their orthogonal trajectories are
$$u - 1 : \frac{\partial \lg \sqrt{G}}{\partial u} = \text{const.}$$
We perceive at once that if
$$G = f(u),$$

i. e. if our surface (Σ) is applicable to a surface of revolution, $R_1 + u = f_1(u)$, and therefore the focal surface cannot be referred to the system of coordinates

$$R_1 + s_\kappa \text{ and } s_\kappa,$$

for the lines

$$R_1 + s_\kappa = \text{const. and } s_\kappa = \text{const.}$$

are identical.

Excluding this case from consideration, we see that since

$$u - \frac{1}{\frac{\partial \lg \sqrt{G}}{\partial u}} = \text{const.} = a,$$

by integrating we obtain

$$\sqrt{G} \,.\, V = u - a,$$

and without any loss of generality we can make $V = 1$. Hence

$$\sqrt{G} = u - a$$

are orthogonal trajectories of the geodesics $u =$ const.

In the case we have excluded from consideration we can introduce a new parameter, namely,

$$\frac{ds_\kappa}{h_\kappa} = dw,$$

where dw is the curvature of the geodesic $v =$ const. Then

$$ds_1^2 = d(s_\kappa + R_1)^2 + R_1^2 \,.\, dw^2,$$

and we see that the focal surface is also applicable to a surface of revolution. This is the well-known Weingarten's proposition.

§10.

We can state without proof, which is altogether similar to that given on page 110, etc., the following theorem:

If the edges of regression of one system of the developable surfaces of the congruence of lines are such that their radii of curvature are functions of the corresponding focal distances, then the second focal surface is applicable to a surface of revolution. We can find all the isometric systems on it by simple quadratures, also can determine all the geodesics which can be drawn on it by simple quadratures. Furthermore, if for different congruences the radii of cur-

vature above mentioned are the same functions of the corresponding focal distances, the second focal surfaces for these congruences are all applicable to one another. If the radii of curvature are functions of the arcs of the edges of regression, the second focal surface is a developable surface.

We give here also without proof the formulæ for direction cosines of the normals to the second focal surface. The proof is based upon the theorem of Koenigs given on page 103. Calling X_2, Y_2, Z_2 these direction cosines, we have

$$X_2 = -\frac{h_\kappa X}{R_\kappa} + \frac{h_\kappa}{T\rho_\kappa}\left[\frac{\partial x}{\partial u}\cdot\frac{F+G\lambda}{\sqrt{E+2F\lambda+G\lambda^2}} - \frac{\partial x}{\partial v}\cdot\frac{E+F\lambda}{\sqrt{E+2F\lambda+G\lambda^2}}\right],$$

$$Y_2 = -\frac{h_\kappa Y}{R_\kappa} + \frac{h_\kappa}{T\rho_\kappa}\left[\frac{\partial y}{\partial u}\cdot\frac{F+G\lambda}{\sqrt{E+2F\lambda+G\lambda^2}} - \frac{\partial y}{\partial v}\cdot\frac{E+F\lambda}{\sqrt{E+2F\lambda+G\lambda^2}}\right],$$

$$Z_2 = -\frac{h_\kappa}{R_\kappa}\cdot Z + \frac{h_\kappa}{T\rho_\kappa}\left[\frac{\partial z}{\partial u}\cdot\frac{F+G\lambda}{\sqrt{E+2F\lambda+G\lambda^2}} - \frac{\partial z}{\partial v}\cdot\frac{E+F\lambda}{\sqrt{E+2F\lambda+G\lambda^2}}\right].$$

Here R_κ is the radius of geodesic curvature of the edge of regression, ρ_κ — radius of principal curvature of the first focal surface, $\lambda = \frac{dv}{du}$ is taken from the differential equation of the lines the tangents to which we consider in the congruence,

$$T^2 = EG - F^2.$$

§11.

Here we want to state the following theorem which escaped us at the time of writing the above matter. Namely, on the page 114 we proved the following theorem: if there exists a relation

$$h_1 = f(R_1),$$

then the lines connecting the centers of geodesic curvature of the u-line with the corresponding centers of principal curvature of the v-line form a congruence of rays normal to a surface. These rays all lie in the tangent planes to the focal surface (S_1), and therefore are all tangent to this surface, i. e. this surface is the surface of the centers of the surface to which all the rays are normal. Now since this surface of the centers is applicable to a surface of revolution, it must be a Weingarten's surface, i. e. its radii of principal curvatures are functions of one another.

The linear element we reduced in this case to the form

$$ds_1'^2 = dv_2^2 + \frac{1}{V_2} \cdot du_1^2.$$

We can take v_2 for the radius ρ_{σ_1} of the surface to which the rays are normal. But

$$v_2 = \int \frac{dv_1}{\sqrt{V_1}} = \int \frac{(1 - \phi^2(R_1'))^{\frac{1}{2}} dR_1'}{\phi(R_1')}$$
$$= F(R_1).$$

Hence $$R_1 = \text{const.}$$

is the line of equal curvature of the above surface, and

$$u_1 = \text{const.}$$

its lines of curvature of one system.

Again, $$u_1 = s_1 + F_1(R_1).$$

If s_1 be a function of R_1, we cannot take $s_1 + R_1$ and R_1 as the parameters of the coordinate lines on the surface (S_1).

But then as before

$$ds_1'^2 = d(s_1 + R_1)^2 + R_1^2 dw^2,$$

where $$dw = \frac{ds_1}{h_1}.$$

In this case the focal surface is applicable to a surface of revolution.

Displacements depending on One, Two and Three Parameters in a Space of Four Dimensions.

By Thomas Craig.

In the present paper I have given as briefly as possible a generalization to a space of four dimensions of the kinematical methods developed by Darboux in the first two volumes of his "Théorie générale des Surfaces." As I have only a very slight knowledge of what has been done in the geometry of a four-dimensional space (Euclidian), I have confined myself entirely to the generalization of Darboux's formulæ, lest otherwise I might merely repeat what is already well known.

We shall first consider a system having one point, O, fixed. Let X, Y, Z, W be the coordinates of a point referred to fixed rectangular axes having O as origin; x, y, z, w the coordinates of the same point referred to moving axes also having O as origin. The following table gives the direction cosines of the two sets of axes referred one to the other:

$$\left.\begin{array}{c|cccc} & X & Y & Z & W \\ \hline x & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ y & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ z & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ w & \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{array}\right\} \qquad (1)$$

Among these 16 cosines we have the following 10 relations:

$$\left.\begin{array}{ll} \alpha_1^2+\beta_1^2+\gamma_1^2+\delta_1^2=1, & \alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2+\delta_1\delta_2=0, \\ \alpha_2^2+\beta_2^2+\gamma_2^2+\delta_2^2=1, & \alpha_1\alpha_3+\beta_1\beta_3+\gamma_1\gamma_3+\delta_1\delta_3=0, \\ \alpha_3^2+\beta_3^2+\gamma_3^2+\delta_3^2=1, & \alpha_1\alpha_4+\beta_1\beta_4+\gamma_1\gamma_4+\delta_1\delta_4=0, \\ \alpha_4^2+\beta_4^2+\gamma_4^2+\delta_4^2=1, & \alpha_2\alpha_3+\beta_2\beta_3+\gamma_2\gamma_3+\delta_2\delta_3=0, \\ & \alpha_2\alpha_4+\beta_2\beta_4+\gamma_2\gamma_4+\delta_2\delta_4=0, \\ & \alpha_3\alpha_4+\beta_3\beta_4+\gamma_3\gamma_4+\delta_3\delta_4=0, \end{array}\right\} \qquad (2)$$

or

$$\left.\begin{array}{ll} \alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2=1, & \alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3+\alpha_4\beta_4=0, \\ \beta_1^2+\beta_2^2+\beta_3^2+\beta_4^2=1, & \alpha_1\gamma_1+\alpha_2\gamma_2+\alpha_3\gamma_3+\alpha_4\gamma_4=0, \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ \delta_1^2+\delta_2^2+\delta_3^2+\delta_4^2=1, & \gamma_1\delta_1+\gamma_2\delta_2+\gamma_3\delta_3+\gamma_4\delta_4=0, \end{array}\right\} \tag{2'}$$

I shall call the figure formed by the four three-dimensional spaces

$$X=0, \quad Y=0, \quad Z=0, \quad W=0, \tag{3}$$

meeting at the point O a *tetrahedroid*, and that formed by the six planes

$$\left.\begin{array}{lll} \begin{cases} X=0, \\ Y=0, \end{cases} & \begin{cases} X=0, \\ Z=0, \end{cases} & \begin{cases} X=0, \\ W=0, \end{cases} \\ \begin{cases} Y=0, \\ Z=0, \end{cases} & \begin{cases} Y=0, \\ W=0, \end{cases} & \begin{cases} Z=0, \\ W=0, \end{cases} \end{array}\right\} \tag{4}$$

a *hexahedron.*

It will be convenient to speak of any such expression as

$$AX+BY+CZ+DW+E=0$$

or

$$ax+by+cz+dw+e=0$$

as a *hyperplane.*

In fact I shall employ the nomenclature used by Poincaré in his memoir "Sur les Résidus des Intégrales doubles," Acta Math., t. 9, pg. 325; thus a *hypersurface* will be expressed by a single relation between the four coordinates of a point, a *surface* by two such relations, a line by three relations.

The following four equations

$$\begin{array}{ll} a & \alpha_1x+\beta_1x+\gamma_1z+\delta_1w=0, \\ b & \alpha_2x+\beta_2y+\gamma_2z+\delta_2w=0, \\ c & \alpha_3x+\beta_3y+\gamma_3z+\delta_3w=0, \\ d & \alpha_4x+\beta_4y+\gamma_4z+\delta_4w=0 \end{array} \tag{5}$$

are the equations of the four hyperplanes forming the faces of the moving tetrahedroid. The planes formed by the combination of these in pairs will be the faces of the hexahedron; these may be denoted as follows:

$$\left.\begin{array}{lll} (a,b)=P^1, & (a,c)=P^2, & (a,d)=P^3, \\ (b,c)=P^4, & (b,d)=P^5, & (c,d)=P^6. \end{array}\right\} \tag{6}$$

To denote the direction cosines of these six planes I employ the notation used by Cole in his paper "On Rotations in Space of Four Dimensions," American

Journal of Mathematics, vol. XII, p. 191. The direction cosines of the plane P^1, for example, will be denoted by

$$P^1_{12},\quad P^1_{13},\quad P^1_{14},\quad P^1_{23},\quad P^1_{24},\quad P^1_{34}.$$

Since our system is an orthogonal one, we have

$$\left.\begin{array}{lll} P^1_{12} = \begin{vmatrix} \alpha_1, & \beta_1 \\ \alpha_2, & \beta_2 \end{vmatrix}, & P^1_{13} = \begin{vmatrix} \alpha_1, & \gamma_1 \\ \alpha_2, & \gamma_2 \end{vmatrix}, & P^1_{14} = \begin{vmatrix} \alpha_1, & \delta_1 \\ \alpha_2, & \delta_2 \end{vmatrix}, \\ P^1_{23} = \begin{vmatrix} \beta_1, & \gamma_1 \\ \beta_2, & \gamma_2 \end{vmatrix}, & P^1_{24} = \begin{vmatrix} \beta_1, & \delta_1 \\ \beta_2, & \delta_2 \end{vmatrix}, & P^1_{34} = \begin{vmatrix} \gamma_1, & \delta_1 \\ \gamma_2, & \delta_2 \end{vmatrix}, \end{array}\right\} \qquad (7)$$

and similar expressions for the 30 direction cosines of the remaining five planes. These quantities P_{ij} satisfy the two equations

$$\left.\begin{array}{ll} P^2_{12} + P^2_{13} + P^2_{14} + P^2_{23} + P^2_{24} + P^2_{34} & = 1, \\ P_{12}P_{34} - P_{13}P_{24} + P_{23}P_{14} & = 0 \end{array}\right\} \qquad (8)$$

(note that $P_{24} = -P_{42}$, so that the second of these can be written in the form

$$P_{12}P_{34} + P_{13}P_{42} + P_{23}P_{14} = 0,$$

the form in which Cole gives it).

I shall denote the moving tetrahedroid by the letter T and the hexahedron by H. To form the equations giving the motion of T we proceed in exactly the same manner as in forming the analogous kinematical equations for space of three dimensions.

Differentiate the second set of equations (2′) and write

$$\begin{aligned} &\sum\beta_i\frac{d\alpha_i}{dt} = -\sum\alpha_i\frac{d\beta_i}{dt} = p_{12}, \\ &\sum\alpha_i\frac{d\gamma_i}{dt} = -\sum\gamma_i\frac{d\alpha_i}{dt} = p_{13}, \\ &\sum\gamma_i\frac{d\beta_i}{dt} = -\sum\beta_i\frac{d\gamma_i}{dt} = p_{23}, \\ &\sum\delta_i\frac{d\alpha_i}{dt} = -\sum\alpha_i\frac{d\delta_i}{dt} = p_{14}, \\ &\sum\delta_i\frac{d\beta_i}{dt} = -\sum\beta_i\frac{d\delta_i}{dt} = p_{24}, \\ &\sum\delta_i\frac{d\gamma_i}{dt} = -\sum\gamma_i\frac{d\delta_i}{dt} = p_{34}, \qquad i = 1, 2, 3, 4. \end{aligned}$$

For $w=0$ the quantities p_{23}, p_{13}, p_{12} are respectively the quantities p, q, r of ordinary space. The extended form of the ordinary kinematical equations can now be written down at once, but instead of writing them for the point (x, y, z, w) I shall at once write them for a point at distance unity from O on each of the axes X, Y, Z, W, referring the reader to Darboux ("Théorie générale des Surfaces," t. I, pg. 4) for the intermediate steps. The equations are

$$
\begin{aligned}
\frac{d\alpha}{dt} &= \qquad\quad + p_{12}\beta - p_{13}\gamma + p_{14}\delta, \\
\frac{d\beta}{dt} &= -p_{12}\alpha \qquad\quad + p_{23}\gamma + p_{24}\delta, \\
\frac{d\gamma}{dt} &= \quad p_{13}\alpha - p_{23}\beta \qquad\quad + p_{34}\delta, \\
\frac{d\delta}{dt} &= -p_{14}\alpha - p_{24}\beta - p_{34}\gamma \qquad\quad .
\end{aligned} \tag{10}
$$

These equations are satisfied by the four sets of direction cosines

$$(\alpha_1, \beta_1, \gamma_1, \delta_1), \quad (\alpha_2, \beta_2, \gamma_2, \delta_2), \quad (\alpha_3, \beta_3, \gamma_3, \delta_3), \quad (\alpha_4, \beta_4, \gamma_4, \delta_4),$$

and these by (2) can be expressed in terms of 6 arbitrary quantities. For the expression of the 16 cosines in terms of 6 arbitrary quantities the reader is referred to a paper by Cayley in vol. XXXII of Crelle and to Cole's paper above cited.

We have here what at first sight seems rather curious, viz. a system of *four* equations of the first order (equations (10)) containing *six* arbitrary constants in their general solution. This is, however, easy to explain. The rotations p_{ij} are the components of rotation about the six planes of the hexahedron H. By aid of (7) and (10) we can form *six* equations giving the derivatives of $P_{12}, P_{13} \ldots . P_{34}$ with respect to t. We have by (7)

$$\frac{dP^1_{12}}{dt} = \frac{d}{dt}(\alpha_1\beta_2 - \alpha_2\beta_1); \tag{11}$$

expanding this and substituting from (10) where we give the quantities $\alpha, \beta, \gamma, \delta$ the suffixes 1 and 2 successively, we arrive at the equation

$$\frac{dP^1_{12}}{dt} = p_{23}P^1_{13} + p_{24}P^1_{14} + p_{13}P^1_{23} - p_{14}P^1_{24}, \tag{12}$$

and similar equations for

$$\frac{dP^1_{13}}{dt}, \quad \frac{dP^1_{14}}{dt}, \quad \frac{dP^1_{23}}{dt}, \quad \frac{dP^1_{24}}{dt}, \quad \frac{dP^1_{34}}{dt}.$$

Dropping the superior affix, we have the following six equations satisfied by the 36 direction cosines P_{ij}^{k}:

$$\left.\begin{aligned}
\frac{dP_{12}}{dt} &= p_{23}P_{13} + p_{24}P_{14} + p_{13}P_{23} - p_{14}P_{24}\\
\frac{dP_{13}}{dt} &= -p_{23}P_{12} + p_{34}P_{14} + p_{12}P_{23} - p_{14}P_{34},\\
\frac{dP_{14}}{dt} &= -p_{24}P_{12} - p_{34}P_{13} + p_{12}P_{24} - p_{13}P_{34},\\
\frac{dP_{23}}{dt} &= -p_{13}P_{12} - p_{12}P_{13} + p_{34}P_{24} - p_{24}P_{34},\\
\frac{dP_{24}}{dt} &= p_{14}P_{12} + p_{12}P_{14} - p_{34}P_{23} + p_{23}P_{34},\\
\frac{dP_{34}}{dt} &= p_{14}P_{13} + p_{13}P_{14} + p_{24}P_{23} - p_{23}P_{24}\ .
\end{aligned}\right\} \tag{13}$$

From these we have at once the two quadratic integrals

$$\left.\begin{aligned}
P_{12}^2 + P_{13}^2 + P_{14}^2 + P_{23}^2 + P_{24}^2 + P_{34}^2 &= \text{const.},\\
P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} &= \text{const.}
\end{aligned}\right\} \tag{14}$$

These last equations show us that by proper substitutions the integration of equations (13) can be conducted to the integration of a system of four equations. The four equations are evidently equations (10).

Equations (10) obviously have the following integral of the second degree:

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \text{const.} \tag{15}$$

If now $(\alpha^0, \beta^0, \gamma^0, \delta^0)$ is a particular integral of (10), we can add the integral of the first degree

$$\alpha\alpha^0 + \beta\beta^0 + \gamma\gamma^0 + \delta\delta^0 = \text{const.} \tag{16}$$

to (15) (Darboux, I, p. 20); and so if we have three particular integrals of (10)

$$(\alpha^0, \beta^0, \gamma^0, \delta^0), \quad (\alpha^1, \beta^1, \gamma^1, \delta^1), \quad (\alpha^2, \beta^2, \gamma^2, \delta^2),$$

we have the following system of equations for the determination of the general integral:

$$\left.\begin{aligned}
\alpha^2 + \beta^2 + \gamma^2 + \delta^2 &= \text{const.},\\
\alpha\alpha^0 + \beta\beta^0 + \gamma\gamma^0 + \delta\delta^0 &= \text{const.},\\
\alpha\alpha^1 + \beta\beta^1 + \gamma\gamma^1 + \delta\delta^1 &= \text{const.},\\
\alpha\alpha^2 + \beta\beta^2 + \gamma\gamma^2 + \delta\delta^2 &= \text{const.},
\end{aligned}\right\} \tag{17}$$

Since $$\alpha^{k^2} + \beta^{k^2} + \gamma^{k^2} + \delta^{k^2} = \text{const.}, \qquad (k = 0, 1, 2) \tag{18}$$

we can join to equations (17) the following:*

$$\Delta = \begin{vmatrix} \alpha, & \beta, & \gamma, & \delta \\ \alpha^0, & \beta^0, & \gamma^0, & \delta^0 \\ \alpha^1, & \beta^1, & \gamma^1, & \delta^1 \\ \alpha^2, & \beta^2, & \gamma^2, & \delta^2 \end{vmatrix} = \text{const.} \tag{19}$$

That $\Delta = \text{const.}$ is obvious from the fact that by (15), (17) and (18) its square is a constant. The last three equations of (17) and equation (19) are linear in $\alpha, \beta, \gamma, \delta$, and so serve to give us the values of these four quantities.

It is only necessary to employ the reasoning on page 6 of Darboux, t. I, and equations (10), (15) and (16), to see that a general solution of (10) involves six arbitrary constants, viz. the six arbitrary constants which serve to define the initial position of the hexahedron H.

Suppose now that the system has no fixed point. We must then introduce the components

$$\xi, \eta, \zeta, \tau$$

of translation of the origin of the moving axes. Here again it is only necessary to reproduce Darboux's reasoning (p. 7 *loc. cit.*) Let X_0, Y_0, Z_0, W_0 denote the coordinates of the moving origin. So far as the *rotation* is concerned, the origin can be fixed, and therefore the 16 direction cosines determined as above. We have now the following equations for the determination of X_0, Y_0, Z_0, W_0:

$$\left.\begin{aligned} \frac{dX_0}{dt} &= \alpha_1\xi + \beta_1\eta + \gamma_1\zeta + \delta_1\tau, \\ \frac{dY_0}{dt} &= \alpha_2\xi + \beta_2\eta + \gamma_2\zeta + \delta_2\tau, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{dW_0}{dt} &= \alpha_4\xi + \beta_4\eta + \gamma_4\zeta + \delta_4\tau, \end{aligned}\right\} \tag{20}$$

* This method of solving equations (17) was, for three dimensions, communicated to me by Professor Echols of the University of Virginia, who had received it from M. E. Cosserat. M. Cosserat's note to Professor Echols was drawn out by a solution which I had published in the Annals of Mathematics of the equations just preceding (6) on page 21 of Darboux's Théorie générale des Surfaces.

The integration of these equations is reduced to quadratures, since the coefficients α_i, β_i, γ_i, δ_i have already been determined as functions of t and ξ, η, ζ, τ, like p_{12}, p_{13} .... p_{34}, are *given* functions of t.

Return now to equations (10) and write

$$\alpha = \frac{2\lambda}{k^2+1}, \quad \beta = \frac{2\mu}{k^2+1}, \quad \gamma = \frac{2\nu}{k^2+1}, \quad \delta = \frac{k^2-1}{k^2+1}, \tag{21}$$

$$k^2 = \lambda^2 + \mu^2 + \nu^2. \tag{22}$$

After some simple reductions we shall find the following equations:

$$\left.\begin{aligned} \frac{d\lambda}{dt} &= \mu p_{12} - \nu p_{13} + \left(\frac{k^2-1}{2}\right) p_{14} - \lambda\,[\lambda p_{14} + \mu p_{24} + \nu p_{34}], \\ \frac{d\mu}{dt} &= -\lambda p_{12} + \nu p_{23} + \left(\frac{k^2-1}{2}\right) p_{24} - \mu\,[\lambda p_{14} + \mu p_{24} + \nu p_{34}], \\ \frac{d\nu}{dt} &= \lambda p_{13} - \mu p_{23} + \left(\frac{k^2-1}{2}\right) p_{34} - \nu\,[\lambda p_{14} + \mu p_{24} + \nu p_{34}]. \end{aligned}\right\} \tag{23}$$

This system of simultaneous equations is, for three unknown functions, a generalization of Riccati's equation. I do not know whether such equations have been studied or not, but, as their integration plays no part in the present paper, it is not necessary to say anything more about them.*

Consider now the case of displacements depending on two parameters, t, u. Let p_{12}, p_{13}, ... p_{34} denote the rotations which depend on t alone; p'_{12}, p'_{13}, ... p'_{34} those which depend on u alone. We have at once the equations

$$\left.\begin{aligned} \frac{\partial\alpha}{\partial t} &= \phantom{-p_{12}\alpha - {}} p_{12}\beta - p_{13}\gamma + p_{14}\delta, \\ \frac{\partial\beta}{\partial t} &= -p_{12}\alpha \phantom{- p_{23}\beta} + p_{23}\gamma + p_{24}\delta, \\ \frac{\partial\gamma}{\partial t} &= p_{13}\alpha - p_{23}\beta \phantom{+ p_{23}\gamma} + p_{34}\delta, \\ \frac{\partial\delta}{\partial t} &= -p_{14}\alpha - p_{24}\beta - p_{34}\gamma \phantom{+ p_{34}\delta}, \end{aligned}\right\} \tag{24}$$

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* A note on the subject of these equations by Mr. John Eiesland will appear in the following number of this Journal.

$$\left.\begin{aligned}
\frac{\partial\alpha}{\partial u} &= \qquad\qquad p'_{12}\beta - p'_{13}\gamma + p'_{14}\delta,\\
\frac{\partial\beta}{\partial u} &= -p'_{12}\alpha \qquad\quad + p'_{23}\gamma + p'_{24}\delta,\\
\frac{\partial\gamma}{\partial u} &= \quad p'_{13}\alpha - p'_{23}\beta \qquad\quad + p'_{34}\delta,\\
\frac{\partial\delta}{\partial u} &= -p'_{14}\alpha - p'_{24}\beta - p'_{34}\gamma \quad .
\end{aligned}\right\} \tag{25}$$

Differentiating each of (24) for u and each of (25) for v and equating the results, we have the following six equations of condition connecting the coefficients p_{ij} and p'_{ij}:

$$\left.\begin{aligned}
\frac{\partial p_{12}}{\partial u} - \frac{\partial p'_{12}}{\partial t} + \begin{vmatrix} p_{13}, & p_{23} \\ p'_{13}, & p'_{23} \end{vmatrix} - \begin{vmatrix} p_{14}, & p_{24} \\ p'_{14}, & p'_{24} \end{vmatrix} &= 0,\\
\frac{\partial p_{13}}{\partial u} - \frac{\partial p'_{13}}{\partial t} + \begin{vmatrix} p_{23}, & p_{12} \\ p'_{23}, & p'_{12} \end{vmatrix} - \begin{vmatrix} p_{34}, & p_{14} \\ p'_{34}, & p'_{14} \end{vmatrix} &= 0,\\
\frac{\partial p_{14}}{\partial u} - \frac{\partial p'_{14}}{\partial t} + \begin{vmatrix} p_{12}, & p_{24} \\ p'_{1}, & p'_{24} \end{vmatrix} - \begin{vmatrix} p_{13}, & p_{34} \\ p'_{13}, & p'_{34} \end{vmatrix} &= 0,\\
\frac{\partial p_{23}}{\partial u} - \frac{\partial p'_{23}}{\partial t} + \begin{vmatrix} p_{12}, & p_{13} \\ p'_{12}, & p'_{13} \end{vmatrix} - \begin{vmatrix} p_{24}, & p_{34} \\ p'_{24}, & p'_{34} \end{vmatrix} &= 0,\\
\frac{\partial p_{24}}{\partial u} - \frac{\partial p'_{24}}{\partial t} + \begin{vmatrix} p_{14}, & p_{12} \\ p'_{14}, & p'_{12} \end{vmatrix} - \begin{vmatrix} p_{34}, & p_{23} \\ p'_{34}, & p'_{23} \end{vmatrix} &= 0,\\
\frac{\partial p_{34}}{\partial u} - \frac{\partial p'_{34}}{\partial t} + \begin{vmatrix} p_{13}, & p_{14} \\ p'_{13}, & p'_{14} \end{vmatrix} - \begin{vmatrix} p_{23}, & p_{24} \\ p'_{23}, & p'_{24} \end{vmatrix} &= 0.
\end{aligned}\right\} \tag{26}$$

We shall arrive at the same set of relations between the quantities p_{ij} and p'_{ij} if we form the two systems of equations similar to (13), one system giving the values of $\frac{\partial P_{ij}}{\partial t}$ and the other giving the values of $\frac{\partial P_{ij}}{\partial u}$. For convenience of reference I shall write the first of each of these systems:

$$\left.\begin{aligned}
\frac{\partial P_{12}}{\partial t} &= p_{23}P_{13} + p_{24}P_{14} + p_{13}P_{23} - p_{14}P_{24},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{27}$$

$$\left.\begin{aligned}
\frac{\partial P_{12}}{\partial u} &= p'_{23}P_{13} + p'_{24}P_{14} + p'_{13}P_{23} - p'_{14}P_{24},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{28}$$

Reciprocally, whenever we have twelve quantities p_{ij}, p'_{ij} satisfying equations (26), there exists a motion in which these twelve quantities are the rotations.

We can use either equations (24) and (25) or (27) and (28) and reproduce almost word for word the reasoning on pages 49–51 of Darboux;* it is not necessary to go over this ground, as the reader can readily supply the missing reasoning.

The question of the integration is in this case led back to the determination of systems of solutions common to two sets of equations formed like (23)—one gives

$$\frac{\partial x}{\partial t}, \quad \frac{\partial y}{\partial t}, \quad \frac{\partial z}{\partial t},$$

and the other

$$\frac{\partial x}{\partial u}, \quad \frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial u}.$$

It is not the purpose of the present paper to take up the study of any of these equations, but merely to indicate what problems require to be solved in generalizing Darboux's methods. If the system has no fixed point, let ξ, η, ζ, τ denote the translations of the origin of T, or H, depending only on t and $\xi', \eta', \zeta', \tau'$, the translations depending only on u; if X_0, Y_0, Z_0, W_0 denote the coordinates of the moving origin, we have for their determination the equations

$$\left.\begin{aligned}\frac{\partial X_0}{\partial t} &= \alpha_1\xi + \beta_1\eta + \gamma_1\zeta + \delta_1\tau, \\ \frac{\partial X_0}{\partial u} &= \alpha_1\xi' + \beta_1\eta' + \gamma_1\zeta' + \delta_1\tau',\end{aligned}\right\} \tag{29}$$

with similar expressions in Y_0, Z_0, W_0. The 16 cosines are, of course, determined just as in the case where the system had one fixed point. To find the conditions to be satisfied by $\xi, \eta, \ldots. \tau'$, differentiate the first of (29) for u and the second for t and equate the results. Since the equations must hold when we replace $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ by $(\alpha_2, \beta_2, \gamma_2, \delta_2)$ or $(\alpha_3, \beta_3, \gamma_3, \delta_3)$ or $(\alpha_4, \beta_4, \gamma_4, \delta_4)$, and since the determinant formed by the 16 cosines is not zero, we have at once the following equations of condition:

$$\left.\begin{aligned}\frac{\partial \xi}{\partial u} - \frac{\partial \xi'}{\partial t} + p_{12}\eta' - p'_{12}\eta - p_{13}\zeta' + p'_{13}\zeta + p_{14}\tau' - p'_{14}\tau &= 0, \\ \frac{\partial \eta}{\partial u} - \frac{\partial \eta'}{\partial t} - p_{12}\xi' + p'_{12}\xi + p_{23}\zeta' - p'_{23}\zeta + p_{24}\tau' - p'_{24}\tau &= 0, \\ \frac{\partial \zeta}{\partial u} - \frac{\partial \zeta'}{\partial t} + p_{13}\xi' - p'_{13}\xi - p_{23}\eta' + p'_{23}\eta + p_{34}\tau' - p'_{34}\tau &= 0, \\ \frac{\partial \tau}{\partial u} - \frac{\partial \tau'}{\partial t} - p_{14}\xi' + p'_{14}\xi - p_{24}\eta' + p'_{24}\eta - p_{34}\tau' + p'_{34}\tau &= 0,\end{aligned}\right\} \tag{30}$$

* It is understood that the references are always to t. I of the "Théorie générale des Surfaces."

Reciprocally, whenever the quantities $\xi \ldots\ldots \tau'$, $p_{12} \ldots\ldots p'_{34}$ satisfy equations (26) and (30), there exists a motion for which these are the translations and rotations; to show this it is only necessary to reproduce the reasoning on page 67 of Darboux; this, however, will not be done.

The following are the projections of the infinitesimal displacements on the axes of x, y, z, w in the case of no fixed point; and when the variables (t, u) are supposed to depend on a single parameter, say s:

$$\left.\begin{aligned}
&dx + \xi dt + \xi' du + (p_{12}dt + p'_{12}du)\,y - (p_{13}dt + p'_{13}du)\,z + (p_{14}dt + p'_{14}du)\,w\,,\\
&dy + \eta dt + \eta' du - (p_{12}dt + p'_{12}du)\,x + (p_{23}dt + p'_{23}du)\,z + (p_{24}dt + p'_{24}du)\,w\,,\\
&dz + \zeta dt + \zeta' du + (p_{13}dt + p'_{13}du)\,x - (p_{23}dt + p'_{23}du)\,y + (p_{34}dt + p'_{34}du)\,w,\\
&dw + \tau dt + \tau' du - (p_{14}dt + p'_{14}du)\,x - (p_{24}dt + p'_{24}du)\,y - (p_{34}dt + p'_{34}du)\,z\,.
\end{aligned}\right\} \quad (31)$$

For the case of a fixed point it is only necessary to make all of the Greek letters zero, and for a one-variable displacement to make the quantities p'_{ij} all zero.

In the case of a one-variable displacement and a curve of triple curvature, the geometrical interpretation of the quantities p_{ij} is given in a Thesis by Mr. J. G. Hardy which will shortly be published. There also will be found a fuller account of some matters which I have merely indicated in what precedes and follows.

In the case of a one-variable displacement we are of course conducted to the geometry of a curve in 4-dimensional space, that is, a locus represented by three equations in x, y, z, w, say

$$\left.\begin{aligned}
\phi_1(x,\ y,\ z,\ w) &= 0\,,\\
\phi_2(x,\ y,\ z,\ w) &= 0\,,\\
\phi_3(x,\ y,\ z,\ w) &= 0\,;
\end{aligned}\right\} \qquad (32)$$

in the case of two-variable displacements we are conducted to the geometry of a surface which may be denoted by the two equations

$$\left.\begin{aligned}
\psi_1(x,\ y,\ z,\ w) &= 0\,,\\
\psi_2(x,\ y,\ z,\ w) &= 0\,.
\end{aligned}\right\} \qquad (33)$$

We can, however, go a step further and consider three-variable displacements when a point moves on (or in?) a curved 3-dimensional space or briefly on a hypersurface. Let t, u, v denote the independent displacement variables in this case. We shall now have the following three sets of equations:

$$\left.\begin{aligned}
\frac{\partial\alpha}{\partial t} &= \qquad\qquad\quad p_{12}\beta - p_{13}\gamma + p_{14}\delta, \\
\frac{\partial\beta}{\partial t} &= -p_{12}\alpha \qquad\qquad + p_{23}\gamma + p_{24}\delta, \\
\frac{\partial\gamma}{\partial t} &= \quad p_{13}\alpha - p_{23}\beta \qquad\quad + p_{34}\delta, \\
\frac{\partial\delta}{\partial t} &= -p_{14}\alpha - p_{24}\beta - p_{34}\gamma \qquad\quad ,
\end{aligned}\right\} \tag{34}$$

$$\left.\begin{aligned}
\frac{\partial\alpha}{\partial u} &= \qquad\qquad\quad p'_{12}\beta - p'_{13}\gamma + p'_{14}\delta, \\
\frac{\partial\beta}{\partial u} &= -p'_{12}\alpha \qquad\qquad + p'_{23}\gamma + p'_{24}\delta, \\
\frac{\partial\gamma}{\partial u} &= \quad p'_{13}\alpha - p'_{23}\beta \qquad\quad + p'_{34}\delta, \\
\frac{\partial\delta}{\partial u} &= -p'_{14}\alpha - p'_{24}\beta - p'_{34}\gamma \qquad\quad ,
\end{aligned}\right\} \tag{34'}$$

$$\left.\begin{aligned}
\frac{\partial\alpha}{\partial v} &= \qquad\qquad\quad p''_{12}\beta - p''_{13}\gamma + p''_{14}\delta, \\
\frac{\partial\beta}{\partial v} &= -p''_{12}\alpha \qquad\qquad + p''_{23}\gamma + p''_{24}\delta, \\
\frac{\partial\gamma}{\partial v} &= \quad p''_{13}\alpha - p''_{23}\beta \qquad\quad + p''_{34}\delta, \\
\frac{\partial\delta}{\partial v} &= -p''_{14}\alpha - p''_{24}\beta - p''_{34}\gamma \qquad\quad .
\end{aligned}\right\} \tag{34''}$$

Three other sets of equations similar to (13) and giving the values of

$$\frac{\partial P_{ij}}{\partial t}, \quad \frac{\partial P_{ij}}{\partial u}, \quad \frac{\partial P_{ij}}{\partial v}$$

can also be written down, but their forms are so obvious that it is not worth while taking up space by reproducing them. We have now to find the conditions which must exist among the quantities p, p' and p'' in order that the three preceding systems of equations may admit of common solutions. These conditions are obtained in the same way as above and are eighteen in number, grouping themselves naturally in six groups of three each, viz.

$$\left.\begin{aligned}
\frac{\partial p_{12}}{\partial u} - \frac{\partial p'_{12}}{\partial t} + \begin{vmatrix} p_{13}, & p_{23} \\ p'_{13}, & p'_{23} \end{vmatrix} - \begin{vmatrix} p_{14}, & p_{24} \\ p'_{14}, & p'_{24} \end{vmatrix} &= 0, \\
\frac{\partial p'_{12}}{\partial v} - \frac{\partial p''_{12}}{\partial u} + \begin{vmatrix} p'_{13}, & p'_{23} \\ p''_{13}, & p''_{23} \end{vmatrix} - \begin{vmatrix} p'_{14}, & p'_{24} \\ p''_{14}, & p''_{24} \end{vmatrix} &= 0, \\
\frac{\partial p''_{12}}{\partial t} - \frac{\partial p_{12}}{\partial v} + \begin{vmatrix} p_{23}, & p_{13} \\ p''_{23}, & p''_{13} \end{vmatrix} - \begin{vmatrix} p_{24}, & p_{14} \\ p'_{24}, & p'_{14} \end{vmatrix} &= 0,
\end{aligned}\right\} \tag{A$_{12}$}$$

$$\left.\begin{aligned}
&\frac{\partial p_{13}}{\partial u}-\frac{\partial p'_{13}}{\partial t}+\begin{vmatrix} p_{23}, & p_{12} \\ p'_{23}, & p_{12} \end{vmatrix}-\begin{vmatrix} p_{34}, & p_{14} \\ p'_{34}, & p_{14} \end{vmatrix}=0, \\
&\frac{\partial p'_{13}}{\partial t}-\frac{\partial p''_{13}}{\partial u}+\begin{vmatrix} p'_{23}, & p'_{12} \\ p''_{23}, & p''_{12} \end{vmatrix}-\begin{vmatrix} p'_{34}, & p'_{1} \\ p''_{34}, & p'_{14} \end{vmatrix}=0, \\
&\frac{\partial p''_{13}}{\partial t}-\frac{\partial p_{13}}{\partial v}+\begin{vmatrix} p''_{23}, & p''_{12} \\ p_{23}, & p_{12} \end{vmatrix}-\begin{vmatrix} p''_{34}, & p''_{14} \\ p_{34}, & p_{14} \end{vmatrix}=0,
\end{aligned}\right\} \qquad (A_{13})$$

$$\left.\begin{aligned}
&\frac{\partial p_{14}}{\partial u}-\frac{\partial p'_{14}}{\partial t}+\begin{vmatrix} p_{12}, & p_{24} \\ p'_{12}, & p'_{24} \end{vmatrix}-\begin{vmatrix} p_{13}, & p_{34} \\ p'_{13}, & p'_{34} \end{vmatrix}=0, \\
&\frac{\partial p'_{14}}{\partial v}-\frac{\partial p''_{14}}{\partial u}+\begin{vmatrix} p'_{12}, & p'_{24} \\ p''_{14}, & p''_{24} \end{vmatrix}-\begin{vmatrix} p'_{13}, & p'_{34} \\ p''_{13}, & p''_{34} \end{vmatrix}=0, \\
&\frac{\partial p''_{14}}{\partial t}-\frac{\partial p_{14}}{\partial v}+\begin{vmatrix} p''_{12}, & p''_{34} \\ p_{14}, & p_{24} \end{vmatrix}-\begin{vmatrix} p''_{13}, & p''_{34} \\ p_{13}, & p_{34} \end{vmatrix}=0.
\end{aligned}\right\} \qquad (A_{14})$$

$$\left.\begin{aligned}
&\frac{\partial p_{23}}{\partial u}-\frac{\partial p'_{23}}{\partial t}+\begin{vmatrix} p_{12}, & p_{13} \\ p'_{12}, & p'_{13} \end{vmatrix}-\begin{vmatrix} p_{24}, & p_{34} \\ p'_{24}, & p'_{34} \end{vmatrix}=0, \\
&\frac{\partial p'_{23}}{\partial v}-\frac{\partial p''_{23}}{\partial u}+\begin{vmatrix} p'_{12}, & p'_{13} \\ p''_{12}, & p''_{13} \end{vmatrix}-\begin{vmatrix} p'_{24}, & p'_{34} \\ p''_{24}, & p''_{34} \end{vmatrix}=0, \\
&\frac{\partial p''_{23}}{\partial t}-\frac{\partial p_{23}}{\partial v}+\begin{vmatrix} p''_{12}, & p''_{13} \\ p_{12}, & p_{13} \end{vmatrix}-\begin{vmatrix} p''_{24}, & p''_{34} \\ p_{24}, & p_{34} \end{vmatrix}=0,
\end{aligned}\right\} \qquad (A_{23})$$

$$\left.\begin{aligned}
&\frac{\partial p_{24}}{\partial u}-\frac{\partial p'_{24}}{\partial t}+\begin{vmatrix} p_{14}, & p_{12} \\ p'_{14}, & p'_{12} \end{vmatrix}-\begin{vmatrix} p_{24}, & p_{23} \\ p'_{24}, & p'_{23} \end{vmatrix}=0, \\
&\frac{\partial p'_{24}}{\partial v}-\frac{\partial p''_{24}}{\partial u}+\begin{vmatrix} p'_{14}, & p'_{12} \\ p''_{14}, & p''_{12} \end{vmatrix}-\begin{vmatrix} p'_{24}, & p'_{23} \\ p''_{24}, & p''_{23} \end{vmatrix}=0, \\
&\frac{\partial p''_{24}}{\partial t}-\frac{\partial p_{24}}{\partial v}+\begin{vmatrix} p''_{14}, & p''_{12} \\ p_{14}, & p_{12} \end{vmatrix}-\begin{vmatrix} p''_{24}, & p''_{23} \\ p_{24}, & p_{23} \end{vmatrix}=0,
\end{aligned}\right\} \qquad (A_{24})$$

$$\left.\begin{aligned}
&\frac{\partial p_{34}}{\partial u}-\frac{\partial p'_{34}}{\partial t}+\begin{vmatrix} p_{13}, & p_{14} \\ p'_{13}, & p'_{14} \end{vmatrix}-\begin{vmatrix} p_{23}, & p_{24} \\ p'_{23}, & p'_{24} \end{vmatrix}=0, \\
&\frac{\partial p'_{34}}{\partial v}-\frac{\partial p''_{34}}{\partial u}+\begin{vmatrix} p'_{13}, & p'_{14} \\ p''_{13}, & p''_{14} \end{vmatrix}-\begin{vmatrix} p'_{23}, & p'_{24} \\ p''_{23}, & p''_{24} \end{vmatrix}=0, \\
&\frac{\partial p''_{34}}{\partial t}-\frac{\partial p_{34}}{\partial v}+\begin{vmatrix} p''_{13}, & p''_{14} \\ p_{13}, & p_{14} \end{vmatrix}-\begin{vmatrix} p''_{23}, & p''_{24} \\ p_{23}, & p_{24} \end{vmatrix}=0,
\end{aligned}\right\} \qquad (A_{34})$$

There are several forms into which the three terms in each group can be thrown,

many of them rather elegant, but inasmuch as the above are really the most convenient forms for applying these conditions, I shall not give any of them.

A final set of conditions must be obtained when we suppose the system to have no fixed point. Let (ξ, η, ζ, τ) denote the translations of the origin of T depending on t alone; $(\xi', \eta', \zeta', \tau')$ those depending on u alone, and $(\xi'', \eta'', \zeta'', \tau'')$ those depending on v alone. If, as before, in equations (29), X_0, Y_0, Z_0, W_0 denote the coordinates of the moving origin, we shall have

$$\left.\begin{aligned} \frac{\partial X_0}{\partial t} &= \alpha_1\xi + \beta_1\eta + \gamma_1\zeta + \delta_1\tau, \\ \frac{\partial X_0}{\partial u} &= \alpha_1\xi' + \beta_1\eta' + \gamma_1\zeta' + \delta_1\tau', \\ \frac{\partial X_0}{\partial v} &= \alpha_1\xi'' + \beta_1\eta'' + \gamma_1\zeta'' + \delta_1\tau''. \end{aligned}\right\} \tag{35}$$

The motion of the tetrahedroid is, of course, obtained just as when one point was fixed, so that quadratures only are necessary to determine $X_0, \ldots . W_0$. The conditions to be satisfied by the $\xi, \eta, \ldots . \tau''$ are readily found and are as follows:

$$\left.\begin{aligned} \frac{\partial\xi}{\partial u} - \frac{\partial\xi'}{\partial t} - \begin{vmatrix} \eta, & \eta' \\ p_{12}, & p'_{12} \end{vmatrix} + \begin{vmatrix} \zeta, & \zeta' \\ p_{13}, & p'_{13} \end{vmatrix} - \begin{vmatrix} \tau, & \tau' \\ p_{14}, & p'_{14} \end{vmatrix} &= 0, \\ \frac{\partial\xi'}{\partial v} - \frac{\partial\xi''}{\partial u} - \begin{vmatrix} \eta', & \eta'' \\ p'_{12}, & p''_{12} \end{vmatrix} + \begin{vmatrix} \zeta', & \zeta'' \\ p', & p''_{13} \end{vmatrix} - \begin{vmatrix} \tau', & \tau'' \\ p'_{14}, & p''_{14} \end{vmatrix} &= 0, \\ \frac{\partial\xi''}{\partial t} - \frac{\partial\xi}{\partial v} - \begin{vmatrix} \eta'', & \eta \\ p''_{12}, & p_{12} \end{vmatrix} + \begin{vmatrix} \zeta'', & \zeta \\ p''_{13}, & p_{13} \end{vmatrix} - \begin{vmatrix} \tau'', & \tau \\ p''_{14}, & p_{14} \end{vmatrix} &= 0, \end{aligned}\right. \tag{ξ}$$

$$\left.\begin{aligned} \frac{\partial\eta}{\partial u} - \frac{\partial\eta'}{\partial t} + \begin{vmatrix} \xi, & \xi' \\ p_{12}, & p'_{12} \end{vmatrix} - \begin{vmatrix} \zeta, & \zeta' \\ p_{23}, & p'_{23} \end{vmatrix} - \begin{vmatrix} \tau, & \tau' \\ p_{24}, & p'_{24} \end{vmatrix} &= 0, \\ \frac{\partial\eta'}{\partial v} - \frac{\partial\eta''}{\partial u} + \begin{vmatrix} \xi', & \xi'' \\ p'_{12}, & p''_{12} \end{vmatrix} - \begin{vmatrix} \zeta', & \zeta'' \\ p'_{23}, & p''_{23} \end{vmatrix} - \begin{vmatrix} \tau', & \tau'' \\ p'_{24}, & p''_{24} \end{vmatrix} &= 0, \\ \frac{\partial\eta''}{\partial t} - \frac{\partial\eta}{\partial v} + \begin{vmatrix} \xi'', & \xi \\ p''_{12}, & p_{12} \end{vmatrix} - \begin{vmatrix} \zeta'', & \zeta \\ p''_{23}, & p_{23} \end{vmatrix} - \begin{vmatrix} \tau'', & \tau \\ p''_{24}, & p_{24} \end{vmatrix} &= 0, \end{aligned}\right. \tag{η}$$

$$\left.\begin{aligned} \frac{\partial\zeta}{\partial u} - \frac{\partial\zeta'}{\partial t} - \begin{vmatrix} \xi, & \xi' \\ p_{13}, & p'_{13} \end{vmatrix} + \begin{vmatrix} \eta, & \eta' \\ p_{23}, & p'_{23} \end{vmatrix} - \begin{vmatrix} \tau, & \tau' \\ p_{34}, & p'_{34} \end{vmatrix} &= 0, \\ \frac{\partial\zeta'}{\partial v} - \frac{\partial\zeta''}{\partial u} - \begin{vmatrix} \xi', & \xi'' \\ p'_{13}, & p''_{13} \end{vmatrix} + \begin{vmatrix} \eta', & \eta'' \\ p'_{23}, & p''_{23} \end{vmatrix} - \begin{vmatrix} \tau', & \tau'' \\ p'_{34}, & p''_{34} \end{vmatrix} &= 0, \\ \frac{\partial\zeta''}{\partial t} - \frac{\partial\zeta}{\partial v} - \begin{vmatrix} \xi'', & \xi \\ p''_{13}, & p_{13} \end{vmatrix} + \begin{vmatrix} \eta'', & \eta \\ p''_{23}, & p_{23} \end{vmatrix} - \begin{vmatrix} \tau'', & \tau \\ p''_{34}, & p_{34} \end{vmatrix} &= 0, \end{aligned}\right. \tag{ζ}$$

$$\left.\begin{aligned}
\frac{\partial\tau}{\partial u}-\frac{\partial\tau'}{\partial t}+\begin{vmatrix}\xi, & \xi'\\ p_{14}, & p'_{14}\end{vmatrix}+\begin{vmatrix}\eta, & \eta'\\ p_{24}, & p'_{24}\end{vmatrix}+\begin{vmatrix}\zeta, & \zeta'\\ p_{34}, & p'_{34}\end{vmatrix}&=0,\\
\frac{\partial\tau'}{\partial v}-\frac{\partial\tau''}{\partial u}+\begin{vmatrix}\xi', & \xi''\\ p'_{14}, & p''_{14}\end{vmatrix}+\begin{vmatrix}\eta', & \eta''\\ p'_{24}, & p''_{24}\end{vmatrix}+\begin{vmatrix}\zeta', & \zeta''\\ p'_{34}, & p''_{34}\end{vmatrix}&=0,\\
\frac{\partial\tau''}{\partial t}-\frac{\partial\tau}{\partial v}+\begin{vmatrix}\xi'', & \xi\\ p''_{14}, & p_{14}\end{vmatrix}+\begin{vmatrix}\eta'', & \eta\\ p''_{24}, & p_{24}\end{vmatrix}+\begin{vmatrix}\zeta'', & \zeta\\ p''_{34}, & p_{34}\end{vmatrix}&=0,
\end{aligned}\right\} \quad (\tau)$$

If t, u, v depend on a single parameter, say s, we have for the projections of the infinitesimal displacements on the four axes of x, y, z, w the following:

$$\left.\begin{aligned}
&dx+\xi dt+\xi' du+\xi'' dv+(p_{12}dt+p'_{12}du+p''_{12}dv)\,y\\
&\qquad -(p_{13}dt+p'_{13}du+p''_{13}dv)\,z+(p_{14}dt+p'_{14}du+p''_{14}dv)\,w,\\
&dy+\eta dt+\eta' du+\eta'' dv-(p_{12}dt+p'_{12}du+p''_{12}dv)\,x\\
&\qquad +(p_{23}dt+p'_{23}du+p''_{23}dv)\,z+(p_{24}dt+p'_{24}du+p''_{24}dv)\,w,\\
&dz+\zeta dt+\zeta' du+\zeta'' dv+(p_{13}dt+p'_{13}du+p''_{13}dv)\,x\\
&\qquad -(p_{23}dt+p'_{23}du+p''_{23}dv)\,y+(p_{34}dt+p'_{34}du+p''_{34}dv)\,w,\\
&dw+\tau dt+\tau' du+\tau'' dv-(p_{14}dt+p'_{14}du+p''_{14}dv)\,x\\
&\qquad -(p_{24}dt+p'_{24}du+p''_{24}dv)\,y-(p_{34}dt+p'_{34}du+p''_{34}dv)\,z.
\end{aligned}\right\} \quad (36)$$

A further hypothesis can be made in this case, viz. that t, u, v depend each on *two* independent parameters, say s and s_1, then we should have

$$\left.\begin{aligned}
dt&=\frac{\partial t}{\partial s}ds+\frac{\partial t}{\partial s_1}ds_1,\\
du&=\frac{\partial u}{\partial s}ds+\frac{\partial u}{\partial s_1}ds_1,\\
dv&=\frac{\partial v}{\partial s}ds+\frac{\partial v}{\partial s_1}ds_1.
\end{aligned}\right\} \quad (37)$$

In this case the t, u, v can be considered as the coordinates of a point on a surface.

In the case of a three-variable displacement we may assume the origin of T to move on a hypersurface, the axis of w to be normal to the hypersurface, and the other three axes to lie in the tangent hyperplane. We shall then have

$$\tau=\tau'=\tau''=0,$$

and the linear element will then be given by

$$ds^2=(\xi dt+\xi' du+\xi'' dv)^2+(\eta dt+\eta' du+\eta'' dv)^2+(\zeta dt+\zeta' du+\zeta'' dv)^2. \quad (38)$$

Write

$$\left.\begin{aligned}\xi^2 + \eta^2 + \zeta^2 &= E_{11},\\ \xi'^2 + \eta'^2 + \zeta'^2 &= E_{22},\\ \xi''^2 + \eta''^2 + \zeta''^2 &= E_{33},\\ \xi\xi' + \eta\eta' + \zeta\zeta' &= F_{12},\\ \xi\xi'' + \eta\eta'' + \zeta\zeta'' &= F_{13},\\ \xi'\xi'' + \eta'\eta'' + \zeta'\zeta'' &= F_{23},\end{aligned}\right\} \tag{39}$$

then

$$ds^2 = E_{11}dt^2 + E_{22}du^2 + E_{33}dv^2 + 2F_{12}dt\,du + 2F_{13}dt\,dv + 2F_{23}du\,dv. \tag{40}$$

The discriminant of this is

$$\Delta^2 = \begin{vmatrix} E_{11} & F_{12} & F_{13} \\ F_{12} & E_{22} & F_{23} \\ F_{13} & F_{23} & E_{33} \end{vmatrix} = \begin{vmatrix} \xi & \eta & \zeta \\ \xi' & \eta' & \zeta' \\ \xi'' & \eta'' & \zeta'' \end{vmatrix}^2. \tag{41}$$

Of course the values of the E's and F's are known when the hypersurface is given, that is, when we know the values of x, y, z, w in terms of t, u, v; in that case we have

$$\left.\begin{aligned} E_{11} &= \sum\left(\frac{\partial x}{\partial t}\right)^2, \quad E_{22} = \sum\left(\frac{\partial x}{\partial u}\right)^2, \quad E_{33} = \sum\left(\frac{\partial x}{\partial v}\right)^2,\\ F_{12} &= \sum\frac{\partial x}{\partial t}\,\frac{\partial x}{\partial u}, \quad F_{13} = \sum\frac{\partial x}{\partial t}\,\frac{\partial x}{\partial v}, \quad F_{23} = \sum\frac{\partial x}{\partial u}\,\frac{\partial x}{\partial v}.\end{aligned}\right\} \tag{42}$$

Equations (39) and (42) serve to determine $\xi, \eta, \ldots. \zeta''$ so far as they can be determined. Any hypothesis concerning the way in which T is *attached* to the hypersurface will give three other conditions which can be joined to (39). We shall then have nine equations which will serve for the complete determination of $\xi, \eta, \ldots. \zeta''$.

Generalizing now the equation given by Darboux, t. II, p. 376 *et seq.*, we have manifestly

$$\left.\begin{aligned} \xi\alpha_1 + \eta\beta_1 + \zeta\gamma_1 &= \frac{\partial x}{\partial t},\\ \xi\alpha_2 + \eta\beta_2 + \zeta\gamma_2 &= \frac{\partial y}{\partial t},\\ \xi\alpha_3 + \eta\beta_3 + \zeta\gamma_3 &= \frac{\partial z}{\partial t},\\ \xi\alpha_4 + \eta\beta_4 + \zeta\gamma_4 &= \frac{\partial w}{\partial t},\end{aligned}\right\} \tag{43}$$

$$\left.\begin{array}{l} \xi'\alpha_1 + \eta'\beta_1 + \zeta'\gamma_1 = \dfrac{\partial x}{\partial u}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \xi'\alpha_4 + \eta'\beta_4 + \zeta'\gamma_4 = \dfrac{\partial w}{\partial u}, \end{array}\right\} \tag{44}$$

$$\left.\begin{array}{l} \xi''\alpha_1 + \eta''\beta_1 + \zeta''\gamma_1 = \dfrac{\partial x}{\partial v}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \xi''\alpha_4 + \eta''\beta_4 + \zeta''\gamma_4 = \dfrac{\partial w}{\partial v}. \end{array}\right\} \tag{45}$$

From these we find at once:

$$\left.\begin{array}{l} \alpha_1 = \dfrac{1}{\Delta}\begin{vmatrix} \dfrac{\partial x}{\partial t} & \eta & \zeta, \\ \dfrac{\partial x}{\partial u} & \eta' & \zeta', \\ \dfrac{\partial x}{\partial v} & \eta'' & \zeta'', \end{vmatrix} \\ \\ \beta_1 = \dfrac{1}{\Delta}\begin{vmatrix} \xi & \dfrac{\partial x}{\partial t} & \zeta, \\ \xi' & \dfrac{\partial x}{\partial u} & \zeta', \\ \zeta'' & \dfrac{\partial x}{\partial v} & \zeta'', \end{vmatrix} \\ \\ \gamma_1 = \dfrac{1}{\Delta}\begin{vmatrix} \xi & \eta & \dfrac{\partial x}{\partial t}, \\ \xi' & \eta' & \dfrac{\partial x}{\partial u}, \\ \xi'' & \eta'' & \dfrac{\partial x}{\partial v}. \end{vmatrix} \end{array}\right\} \tag{46}$$

The quantities $(\alpha_2, \beta_2, \gamma_2)$ will be got by changing x into y in these last equations; $(\alpha_3, \beta_3, \gamma_3)$ by changing x into z, and $(\alpha_4, \beta_4, \gamma_4)$ by changing x into w. To get the cosines δ_1, δ_2, δ_3, δ_4 we need only to recall the properties of the orthogonal substitution for which $\alpha_1, \ldots \delta_4$ are the coefficients. Or we may use the equations which express that the normal whose direction cosines are δ_1, δ_2, δ_3, δ_4 is at right angles to the tangent hyperplane; these are

$$\left.\begin{aligned}\delta_1\frac{\partial x}{\partial t}+\delta_2\frac{\partial y}{\partial t}+\delta_3\frac{\partial z}{\partial t}+\delta_4\frac{\partial w}{\partial t}=0,\\ \delta_1\frac{\partial x}{\partial u}+\delta_2\frac{\partial y}{\partial u}+\delta_3\frac{\partial z}{\partial u}+\delta_4\frac{\partial w}{\partial u}=0,\\ \delta_1\frac{\partial x}{\partial v}+\delta_2\frac{\partial y}{\partial v}+\delta_3\frac{\partial z}{\partial v}+\delta_4\frac{\partial w}{\partial v}=0.\end{aligned}\right\}\tag{47}$$

We shall have by either method

$$\left.\begin{aligned}\delta_1=\frac{1}{\Delta}\,\frac{\partial(y,z,w)}{\partial(t,u,v)},\quad \delta_2=-\frac{1}{\Delta}\,\frac{\partial(z,w,x)}{\partial(t,u,v)},\\ \delta_3=\frac{1}{\Delta}\,\frac{\partial(w,x,y)}{\partial(t,u,v)},\quad \delta_4=-\frac{1}{\Delta}\,\frac{\partial(x,y,z)}{\partial(t,u,v)}.\end{aligned}\right\}\tag{48}$$

The direction cosines P_{ij}^k, thirty-six in number, of the six coordinate planes are readily found when the 16 cosines α_i, β_i, γ_i, δ_i $(i=1,2,3,4)$ are known.

To obtain the rotations p_{ij}, p'_{ij}, p''_{ij}, we proceed as on p. 378, t. II of Darboux, using equations (43), (44), (45) and (48). We have at once the following six relations:

$$\left.\begin{aligned}\sum\delta_1\,d\,\frac{\partial x}{\partial t}=\xi\,(p_{14}dt+p'_{14}du+p''_{14}dv)\\ +\eta\,(p_{24}dt+p'_{24}du+p''_{24}dv)\\ +\zeta\,(p_{34}dt+p'_{34}du+p''_{34}dv),\end{aligned}\right\}\tag{49}$$

$$\left.\begin{aligned}\sum\delta_1\,d\,\frac{\partial x}{\partial u}=\xi'\,(p_{14}dt+p'_{14}du+p''_{14}dv),\\ +\eta'\,(p_{24}dt+p'_{24}du+p''_{24}dv),\\ +\zeta'\,(p_{34}dt+p'_{34}du+p''_{34}dv),\end{aligned}\right\}\tag{50}$$

$$\left.\begin{aligned}\sum\delta_1\,d\,\frac{\partial x}{\partial v}=\xi''\,(p_{14}dt+p'_{14}du+p''_{14}dv),\\ +\eta''\,(p_{24}dt+p'_{24}du+p''_{24}dv),\\ +\zeta''\,(p_{34}dt+p'_{34}du+p''_{34}dv),\end{aligned}\right\}\tag{51}$$

$$\left.\begin{aligned}\sum\frac{\partial x}{\partial t}\,d\,\frac{\partial x}{\partial u}=\xi d\xi'+\eta d\eta'+\zeta d\zeta'+(\xi'\eta-\xi\eta')(p_{12}dt+p'_{12}du+p''_{12}dv)\\ +(\xi\zeta'-\xi'\zeta)(p_{13}dt+p'_{13}du+p''_{13}dv)\\ +(\eta'\zeta-\eta\zeta')(p_{23}dt+p'_{23}du+p''_{23}dv),\end{aligned}\right\}\tag{52}$$

$$\left.\begin{aligned}\sum\frac{\partial x}{\partial u}\,d\,\frac{\partial x}{\partial v}=\xi'd\xi''+\eta'd\eta''+\zeta'd\zeta''+(\xi''\eta'-\xi'\eta'')(p_{12}dt+p'_{12}du+p''_{12}dv)\\ +(\xi'\zeta''-\xi''\zeta')(p_{13}dt+p'_{13}du+p''_{13}dv)\\ +(\eta''\zeta'-\eta'\zeta'')(p_{23}dt+p'_{23}du+p''_{23}dv),\end{aligned}\right\}\tag{53}$$

$$\left.\begin{aligned}\sum\frac{\partial x}{\partial v}\, d\,\frac{\partial x}{\partial t} = \xi'' d\zeta + \eta'' d\eta + \zeta'' d\zeta &+ (\xi\eta'' - \zeta''\eta)(p_{12}dt + p'_{12}du + p''_{12}dv)\\ &+ (\xi''\zeta - \xi\zeta'')(p_{13}dt + p'_{13}du + p''_{13}dv)\\ &+ (\eta\zeta'' - \eta''\zeta)(p_{23}dt + p'_{23}du + p''_{23}dv)\,.\end{aligned}\right\}\quad(54)$$

Define now the six determinants

$$D_{11},\ D_{22},\ D_{33},\ D_{12},\ D_{13},\ D_{23}$$

by the equation

$$\left.\begin{aligned}&D_{11}dt^2 + D_{22}du^2 + D_{33}dv^2 + 2D_{12}dt\,du + 2D_{13}dt\,dv + 2D_{23}du\,dv\\ &\quad = \frac{1}{\Delta}\left|\begin{aligned}&\frac{\partial x}{\partial t},\ \frac{\partial x}{\partial u},\ \frac{\partial x}{\partial v},\ \frac{\partial^2 x}{\partial t^2}dt^2 + \frac{\partial^2 x}{\partial u^2}du^2 + \frac{\partial^2 x}{\partial v^2}dv^2\\ &\quad + 2\frac{\partial^2 x}{\partial t\,\partial u}dt\,du + 2\frac{\partial^2 x}{\partial t\,\partial v}dt\,dv + 2\frac{\partial^2 x}{\partial u\,\partial v}du\,dv,\end{aligned}\right|\end{aligned}\right\}\quad(55)$$

only one line of the determinant on the right-hand side is written, the others are, of course, obtained by changing x into y, z and w successively.

Introducing the values of δ_1, δ_2, δ_3, δ_4, we have

$$\left.\begin{aligned}D_{11} &= \sum\delta_1\frac{\partial^2 x}{\partial t^2}\ , & D_{22} &= \sum\delta_1\frac{\partial^2 x}{\partial u^2}\ , & D_{33} &= \sum\delta_1\frac{\partial^2 x}{\partial v^2}\ ,\\ D_{12} &= \sum\delta_1\frac{\partial^2 x}{\partial t\,\partial u}, & D_{13} &= \sum\delta_1\frac{\partial^2 x}{\partial t\,\partial v}, & D_{23} &= \sum\delta_1\frac{\partial^2 x}{\partial u\,\partial v}\,.\end{aligned}\right\}\quad(56)$$

Equations (44), (50) and (51) can now be written

$$\left.\begin{aligned}D_{11}dt + D_{12}du + D_{13}dv &= \xi P_{14} + \eta P_{24} + \zeta P_{34}\,,\\ D_{12}dt + D_{22}du + D_{23}dv &= \xi' P_{14} + \eta' P_{24} + \zeta' P_{34}\,,\\ D_{13}dt + D_{23}du + D_{33}dv &= \xi'' P_{14} + \eta'' P_{24} + \zeta'' P_{34}\,.\end{aligned}\right\}\quad(57)$$

The P_{14}, P_{24}, P_{34} are abbreviations whose definitions are obvious. We have now

$$\Delta P_{14} = \begin{vmatrix} D_{11}dt + D_{12}du + D_{13}dv, & \eta & \zeta\,, \\ D_{12}dt + D_{22}du + D_{23}dv, & \eta' & \zeta', \\ D_{13}dt + D_{23}du + D_{33}dv, & \eta'' & \zeta'', \end{vmatrix}\quad(58)$$

or substituting for P_{14} its value,

$$\left.\begin{aligned}&\Delta\,(p_{14}dt + p'_{14}du + p''_{14}dv) =\\ &\quad\begin{vmatrix} D_{11} & \eta & \zeta \\ D_{12} & \eta' & \zeta' \\ D_{13} & \eta'' & \zeta'' \end{vmatrix} dt + \begin{vmatrix} D_{12} & \eta & \zeta \\ D_{22} & \eta' & \zeta' \\ D_{23} & \eta'' & \zeta'' \end{vmatrix} du + \begin{vmatrix} D_{13} & \eta & \zeta \\ D_{23} & \eta' & \zeta' \\ D_{33} & \eta'' & \zeta'' \end{vmatrix} dv\,.\end{aligned}\right\}\quad(59)$$

Similarly,

$$\left.\begin{array}{l}\Delta(p_{24}dt+p'_{24}du+p''_{24}dv)= \\ \qquad\begin{vmatrix}\xi & D_{11} & \zeta \\ \xi' & D_{12} & \zeta' \\ \xi'' & D_{13} & \zeta''\end{vmatrix}dt+\begin{vmatrix}\xi & D_{12} & \zeta \\ \xi' & D_{22} & \zeta' \\ \xi'' & D_{23} & \zeta''\end{vmatrix}du+\begin{vmatrix}\xi & D_{13} & \zeta \\ \xi' & D_{23} & \zeta' \\ \xi'' & D_{33} & \zeta''\end{vmatrix}dv\end{array}\right\}\quad(60)$$

and

$$\left.\begin{array}{l}\Delta(p_{34}dt+p'_{34}du+p''_{34}dv)= \\ \qquad\begin{vmatrix}\xi & \eta & D_{11} \\ \xi' & \eta' & D_{12} \\ \xi'' & \eta'' & D_{13}\end{vmatrix}dt+\begin{vmatrix}\xi & \eta & D_{12} \\ \xi' & \eta' & D_{22} \\ \xi'' & \eta'' & D_{23}\end{vmatrix}du+\begin{vmatrix}\xi & \eta & D_{13} \\ \xi' & \eta' & D_{23} \\ \xi'' & \eta'' & D_{33}\end{vmatrix}dv.\end{array}\right\}\quad(61)$$

It is only necessary in each of these last three equations to equate separately the coefficients of dt, du, dv and so obtain the values of the nine rotations

$$\begin{array}{lll} p_{14}, & p'_{14}, & p''_{14}, \\ p_{24} & p'_{24}, & p''_{24}, \\ p_{34}, & p'_{34}, & p''_{34}. \end{array}$$

To get the remaining nine rotations $p_{12}, p'_{12}, \ldots . p''_{23}$ we use equations (52), (53) and (54).

We have first to calculate the values of the left-hand members of these equations.

From (52) we have

$$\left.\begin{array}{l}\sum\frac{\partial x}{\partial t}\,d\,\frac{\partial x}{\partial u}=dt\sum\frac{\partial x}{\partial t}\,\frac{\partial^2 x}{\partial t\,\partial u}+du\sum\frac{\partial x}{\partial t}\,\frac{\partial^2 x}{\partial u^2}+dv\sum\frac{\partial x}{\partial t}\,\frac{\partial^2 x}{\partial u\,\partial v} \\ \qquad=\frac{1}{2}\,\frac{\partial E_{11}}{\partial u}\,dt+\left[\frac{\partial F_{12}}{\partial u}-\frac{1}{2}\,\frac{\partial E_{22}}{\partial t}\right]du+\frac{1}{2}\left[\frac{\partial F_{13}}{\partial u}+\frac{\partial F_{12}}{\partial v}-\frac{\partial F_{23}}{\partial t}\right]dv.\end{array}\right\}\quad(62)$$

From (53) we have

$$\left.\begin{array}{l}\sum\frac{\partial x}{\partial u}\,d\,\frac{\partial x}{\partial v}=dt\sum\frac{\partial x}{\partial u}\,\frac{\partial^2 x}{\partial t\,\partial v}+du\sum\frac{\partial x}{\partial u}\,\frac{\partial^2 x}{\partial u\,\partial v}+dv\sum\frac{\partial x}{\partial u}\,\frac{\partial^2 x}{\partial v^2} \\ \qquad=\frac{1}{2}\left[\frac{\partial F_{12}}{\partial v}-\frac{\partial F_{13}}{\partial u}+\frac{\partial F_{23}}{\partial t}\right]dt+\frac{1}{2}\,\frac{\partial E_{22}}{\partial v}\,du+\left[\frac{\partial F_{13}}{\partial v}-\frac{1}{2}\,\frac{\partial E_{33}}{\partial u}\right]dv;\end{array}\right\}\quad(63)$$

and from (54) we get

$$\left.\begin{array}{l}\sum\frac{\partial x}{\partial v}\,d\,\frac{\partial x}{\partial t}=dt\sum\frac{\partial x}{\partial v}\,\frac{\partial^2 x}{\partial t^2}+du\sum\frac{\partial x}{\partial v}\,\frac{\partial^2 x}{\partial u\,\partial t}+dv\sum\frac{\partial x}{\partial v}\,\frac{\partial^2 x}{\partial t\,\partial v} \\ \qquad=\left[\frac{\partial F_{13}}{\partial t}-\frac{1}{2}\,\frac{\partial E_{11}}{\partial v}\right]dt+\frac{1}{2}\left[\frac{\partial F_{13}}{\partial u}-\frac{\partial F_{12}}{\partial v}+\frac{\partial F_{23}}{\partial t}\right]du+\frac{1}{2}\,\frac{\partial E_{33}}{\partial t}\,dv.\end{array}\right\}\quad(64)$$

For brevity, write these last three equations in the form

$$\sum \frac{\partial x}{\partial t}\, d\,\frac{\partial x}{\partial u} = A_1 dt + B_1 du + C_1 dv, \tag{62'}$$

$$\sum \frac{\partial x}{\partial u}\, d\,\frac{\partial x}{\partial v} = A_2 dt + B_2 du + C_2 dv, \tag{63'}$$

$$\sum \frac{\partial x}{\partial v}\, d\,\frac{\partial x}{\partial t} = A_3 dt + B_3 du + C_3 dv. \tag{64'}$$

The values of the A_i, B_i, C_i are seen at once by comparing the equations (62), (63), (64) with (62'), (63'), (64').

Write also for brevity

$$\left.\begin{aligned} P_{12} &= p_{12}dt + p'_{12}du + p''_{12}dv, \\ P_{13} &= p_{13}dt + p'_{13}du + p''_{13}dv, \\ P_{23} &= p_{23}dt + p'_{23}du + p''_{23}dv, \end{aligned}\right\} \tag{65}$$

Equations (52), (53) and (54) can now be put in the following forms:

$$\left.\begin{aligned} &\left[A_1 - \xi\frac{\partial \xi'}{\partial t} - \eta\frac{\partial \eta'}{\partial t} - \zeta\frac{\partial \zeta'}{\partial t}\right] dt + \left[B_1 - \xi\frac{\partial \xi'}{\partial u} - \eta\frac{\partial \eta'}{\partial u} - \zeta\frac{\partial \zeta'}{\partial u}\right] du \\ &\qquad + \left[C_1 - \xi\frac{\partial \xi'}{\partial v} - \eta\frac{\partial \eta'}{\partial v} - \zeta\frac{\partial \zeta'}{\partial v}\right] dv \\ &= -(\xi\eta' - \xi'\eta)\,P_{12} - (\zeta\xi' - \zeta'\xi)\,P_{13} - (\eta\zeta' - \eta'\zeta)\,P_{23}, \end{aligned}\right\} \tag{52'}$$

$$\left.\begin{aligned} &\left[A_2 - \xi'\frac{\partial \xi''}{\partial t} - \eta'\frac{\partial \eta''}{\partial t} - \zeta'\frac{\partial \zeta''}{\partial t}\right] dt + \left[B_2 - \xi'\frac{\partial \xi''}{\partial u} - \eta'\frac{\partial \eta''}{\partial u} - \zeta'\frac{\partial \zeta''}{\partial u}\right] du \\ &\qquad + \left[C_2 - \xi'\frac{\partial \xi''}{\partial v} - \eta'\frac{\partial \eta''}{\partial v} - \zeta'\frac{\partial \zeta''}{\partial v}\right] dv \\ &= -(\xi'\eta'' - \xi''\eta')\,P_{12} - (\xi''\zeta' - \xi'\zeta'')\,P_{13} - (\eta'\zeta'' - \eta''\zeta')\,P_{23}, \end{aligned}\right\} \tag{53'}$$

$$\left.\begin{aligned} &\left[A_3 - \xi''\frac{\partial \xi}{\partial t} - \eta''\frac{\partial \eta}{\partial t} - \zeta''\frac{\partial \zeta}{\partial t}\right] dt + \left[B_3 - \xi''\frac{\partial \xi}{\partial u} - \eta''\frac{\partial \eta}{\partial u} - \zeta''\frac{\partial \zeta}{\partial u}\right] du \\ &\qquad + \left[C_3 - \xi''\frac{\partial \xi}{\partial v} - \eta''\frac{\partial \eta}{\partial v} - \zeta''\frac{\partial \zeta}{\partial v}\right] dv \\ &= -(\xi''\eta - \xi\eta'')\,P_{12} - (\xi\zeta'' - \xi''\zeta)\,P_{13} - \eta''\zeta - (\eta''\zeta - \eta\zeta'')\,P_{23}. \end{aligned}\right\} \tag{54'}$$

The determinant of the right-hand sides of these equations is obviously

$$-\Delta^2 = -\begin{vmatrix} \xi, & \eta, & \zeta \\ \xi', & \eta', & \zeta' \\ \xi'', & \eta'', & \zeta'' \end{vmatrix}^2 = -\begin{vmatrix} E_{11}, & F_{12}, & F_{13}, \\ F_{12}, & E_{22}, & F_{23}, \\ F_{13}, & F_{23}, & E_{33}. \end{vmatrix}.$$

We have now only to solve these last three equations in order to obtain the values of P_{12}, P_{13}, P_{23}, then replacing these quantities by their values from (65) and equating separately the coefficients of dt, du, dv on each side of the results, we shall have the values of the nine rotations

$$\begin{matrix} p_{12}, & p'_{12}, & p''_{12}, \\ p_{13}, & p'_{13}, & p''_{13}, \\ p_{23}, & p'_{23}, & p''_{23}, \end{matrix}$$

which, it will be noted, depend only on the linear element. The work of calculation is somewhat long, but perfectly simple. We find

$$\left.\begin{aligned} P_{12} &= p_{12}dt + p'_{12}du + p''_{12}dv \\ &= -\frac{1}{\Delta}\Big\{\xi''\Big[\Big(A_1-\sum\xi\frac{\partial\xi'}{\partial t}\Big)dt+\Big(B_1-\sum\xi\frac{\partial\xi'}{\partial u}\Big)du+\Big(C_1-\sum\xi\frac{\partial\xi'}{\partial v}\Big)dv\Big] \\ &\qquad +\xi\Big[\Big(A_2-\sum\xi'\frac{\partial\xi''}{\partial t}\Big)dt+\Big(B_2-\sum\xi'\frac{\partial\xi''}{\partial u}\Big)du+\Big(C_2-\sum\xi'\frac{\partial\xi''}{\partial v}\Big)dv\Big] \\ &\qquad +\xi\Big[\Big(A_3-\sum\xi''\frac{\partial\xi}{\partial t}\Big)dt+\Big(B_3-\sum\xi''\frac{\partial\xi}{\partial u}\Big)du+\Big(C_3-\sum\xi''\frac{\partial\xi}{\partial v}\Big)dv\Big]\Big\} \end{aligned}\right\} \quad (66)$$

We shall have P_{13} and P_{23} by changing the multipliers ξ'', ξ, ξ' of the brackets into η'', η, η' and ζ'', ζ, ζ' respectively. If we take the simplest case, viz. the one corresponding to lines of curvature on a surface in three-dimensional space, we shall have

$$\left.\begin{aligned} D_{12} = D_{13} = D_{23} = 0, \\ F_{12} = F_{13} = F_{23} = 0. \end{aligned}\right\} \quad (67)$$

It is easy to see that these last conditions involve the following:

$$\xi' = \xi'' = \eta = \eta'' = \zeta = \zeta' = 0,$$

and so

$$\left.\begin{aligned} &\xi = \sqrt{E_{11}}, \quad \eta' = \sqrt{E_{22}}, \quad \zeta'' = \sqrt{E_{33}}, \\ &\Delta = \xi\eta'\zeta'' = \sqrt{E_{11}E_{22}E_{33}}. \end{aligned}\right\} \quad (69)$$

From equations (59), (60) and (61) we now get

$$\left.\begin{matrix} p_{14} = \dfrac{D_{11}}{\sqrt{E_{11}}}, & p'_{14} = 0 \quad, & p''_{14} = 0 \quad, \\ p_{24} = 0 \quad, & p'_{24} = \dfrac{D_{22}}{\sqrt{E_{22}}}, & p''_{24} = 0 \quad, \\ p_{34} = 0 \quad, & p'_{34} = 0 \quad, & p''_{34} = \dfrac{D_{33}}{\sqrt{E_{33}}}. \end{matrix}\right\} \quad (70)$$

These give us, of course, expressions for the three principal radii of curvature at a point of the hypersurface where the three parametric surfaces

$$t = \text{const.}, \quad u = \text{const.}, \quad v = \text{const.}$$

are *surfaces of curvature.*

We have also from (66) and the two similar equations derived from it by changing ξ into η and into ζ respectively:

$$\left.\begin{array}{lll} p_{12} = 0 \qquad , & p'_{12} = \dfrac{-1}{2\sqrt{E_{22}E_{33}}}\dfrac{\partial E_{22}}{\partial v}, & p''_{12} = \dfrac{1}{2\sqrt{E_{22}E_{33}}}\dfrac{\partial E_{33}}{\partial u}, \\ p_{13} = \dfrac{1}{2\sqrt{E_{11}E_{33}}}\dfrac{\partial E_{11}}{\partial v}, & p'_{13} = 0 \qquad , & p''_{13} = \dfrac{-1}{2\sqrt{E_{11}E_{33}}}\dfrac{\partial E_{33}}{\partial t}, \\ p_{23} = \dfrac{-1}{2\sqrt{E_{11}E_{22}}}\dfrac{\partial E_{11}}{\partial u}, & p'_{23} = \dfrac{1}{2\sqrt{E_{11}E_{22}}}\dfrac{\partial E_{12}}{\partial t}, & p''_{23} = 0 \qquad . \end{array}\right\} \quad (71)$$

These formulæ are analogous to those for the radii of geodesic curvature of the lines of curvature in ordinary space.

It is obvious that an enormous number of formulæ concerning hypersurfaces, surfaces and curves in a four-dimensional space, can be derived from the preceding brief generalization of Darboux's methods. A problem of particular interest, as it seems to me, is the case where the three variables t, u, v depend on two independent parameters; for the study of this problem Poincaré's memoir ("Sur les Résidus des Intégrales doubles," Acta Math., t. IX) and the treatise "Théorie des Fonctions Algébriques de deux Variables indépendantes," by Picard and Simart are full of suggestions.

BALTIMORE, *Dec.* 2, 1897.

*Further Researches in the Theory of Quintic Equations.**

By Emory McClintock.

1. This paper comprises in substance four successive parts: first, a preliminary classification of quintics between reducible and irreducible, and again between resolvable and unresolvable (paragraphs 2–6); secondly, a simplified restatement of my earlier discoveries (7–16); thirdly, the presentation of the necessary form of the coefficients of the general resolvable quintic (17–32); and lastly, the development of a theorem according to which any given resolvable quintic engenders another for which my sextic resolvent has the same rational value (33–42). In the course of the first part, a method is presented (4–5) for detecting the rational factors of reducible quintics, a method which is applicable as well to equations of other degrees; and this is followed (6) by a method for recognizing quintics which are unresolvable because of their having two and only two imaginary roots. The second part recalls my paper of 1885 entitled "Analysis of Quintic Equations," which was published in the American Journal of Mathematics, vol. VIII, pp. 45–84.† In that paper I showed that there are three cyclic functions of the roots, functions which have a rational value when the quintic is resolvable, namely, t, v, s, connected by the relation $s = t^2 v$, two of which must be taken into account in any simple discussion of the resolution of the quintic. Regarding the recognition of these quantities as my most important contribution to the development of the subject, I dwelt repeatedly upon their usefulness, and gave particulars of earlier investigations which had failed either of success or of simplicity for want of these necessary auxiliaries.

* Read at the Toronto meeting of the American Mathematical Society, August 17, 1897, when copies of a printed sheet containing the formulæ were supplied, for convenience, to the members present.

† I take this opportunity to point out certain *errata:* p. 47, in (5) and (10), $\frac{1}{15}$ is blurred; p. 51, in (32), for $u^{\frac{2}{3}}$ read $u^{\frac{3}{2}}$; p. 52, in (34), for y read w; last line of p. 59 and second of p. 60, for $\phi\psi^{-1}$ read $\psi\phi^{-1}$; p. 60, in (47), insert ϕ after (5D); p. 60, line 8 from bottom, for 2 read 3; p. 68, line 15 from bottom, for 16 read -16; p. 73, in (106), for d^6 read d_6; p. 75, line 11 from bottom, insert v^{-2} after p^2; next line, for p^2v^{-2} read p^2v^{-4}; next line, for pv^{-1} read pv^{-2}.

Starting from the well-known theory of Bezout and Euler, who assigned four elements, say u_1, u_2, u_3, u_4, as functions of the roots, I developed two chief auxiliary equations containing only t and v in addition to the coefficients, from which by elimination I produced two sextic resolvents, one in t, which was new, the other in v, which was a simplified reproduction of the only resolvent previously known, that of Malfatti. I also showed how a resolvent in s was readily derivable from that in v, and developed formulæ for determining the other two of the quantities t, v, s, whenever one of them became known by means of a resolvent. Finally, I supplied formulæ for determining the roots of the quintic from ascertained values of t and v. In the second part (7–16) of the present paper, besides simplifying one of the formulæ last mentioned, I reproduce much of the work referred to, but in a different order, which appears to reduce the algebraic labor to a minimum. In fact, I adopt a new method which might be applied to equations of other degrees, and which is the precise reverse of that of Bezout and Euler: instead of defining the elements as functions of the roots, I start with the elements and define the various quantities with which I deal, including the coefficients and the unknown quantity itself, as functions of the elements. In the third part, I develop (17–20) formulæ by which, assigning rational values at will to four parameters, we are enabled to produce the coefficients, and the quantities t and v, for all possible resolvable quintics; I consider (21–25) the modifications of this system which become necessary in critical cases, remark (26) upon the difficulty of constructing resolvable quintics of the form $y^5 + 10\gamma y^3 + 10\delta y^2 + \zeta = 0$, and give reasons (27) why simpler parameters cannot be devised. After remarking (28) that in general there are four conjugate quintics for which t and v have identical values, I refer (29–32) to the history of previous partial solutions of this problem of constructing resolvable quintics. The rest of the paper (33–41) contains the proof of, and some comments upon, the fact already intimated, that if my resolvent sextic be found to have a rational root, and if the sextic be reduced to a quintic by a division depriving it of the rational root in question, the resolvent of the new quintic will itself have the same rational root.

2. The general quintic is

$$ax^5 + 5bx^4 + 10cx^3 + 10dx^2 + 5ex + f = 0, \tag{1}$$

which, if $x = y - ba^{-1}$, takes the shorter form

$$y^5 + 10\gamma y^3 + 10\delta y^2 + 5\varepsilon y + \zeta = 0, \tag{2}$$

$$\left.\begin{aligned}\text{where}\qquad \gamma &= a^{-2}(ac-b^2),\\ \delta &= a^{-3}(a^2d-3abc+2b^3),\\ \varepsilon &= a^{-4}(a^3e-4a^2bd+6ab^2c-3b^4),\\ \zeta &= a^{-5}(a^4f-5a^3be+10a^2b^2d-10ab^3c+4b^5).\end{aligned}\right\}\qquad(3)$$

We shall assume the coefficients to be rational, recollecting however that what we find true for rational coefficients must also be true for irrational coefficients, provided that the word "rational," wherever used, be so extended in meaning as to comprise the coefficients as well as all rational numbers, embracing the whole in one hypothetical domain of rationality. It is known that the general quintic is not solvable algebraically, that is to say, by the simple algebraic processes of addition, subtraction, multiplication, division, the raising of powers, and the extraction of roots. Some quintics are so solvable. Those which can be broken up into factors directly are called reducible. Those which can be broken up indirectly into five linear factors, through the determination of the several roots by the aid of a sextic resolvent, may be classed as resolvable. There are thus four classes of quintics: the resolvable-reducible class, which includes, with some others, those having five rational roots; the unresolvable-reducible class; the resolvable-irreducible class; and the unresolvable-irreducible class, the last not being solvable by radicals. We shall here have chiefly to do with resolvable equations, but a few preliminary words concerning the criteria of reducibility will not be out of place.

3. A quintic in y is reducible when it is divisible either by $y+m$ or by y^2+py+q. We may proceed first to test a quintic by trying to find a linear factor, failing which we may look for a quadratic factor. In this it will be assumed that the coefficient of y^5 is unity. (If it has any other positive integral value a, the differences hereafter spoken of will all be multiplied by a or by some factor of a.) If we have the equation, for example, $y^5-5y-3=0$, the usual test for a linear factor is to substitute successively for y its possible rational values, namely, $1,\ -1,\ 3,\ -3$. It is better, however, to pursue another known method, by which at first only two values are substituted, 1 and -1. The given equation being $\phi(y)=0$, with coefficients made integral, we set down in order the numerical values of $\phi(-1)$, $\phi(0)$, $\phi(1)$, and inspect them to see whether they respectively possess integral divisors ascending in arithmetical progression, the common difference being 1. In the example

$y^5-5y-3=0$, the three values are $\phi(-1)=1$, $\phi(0)=-3$, $\phi(1)=-7$. These obviously do not possess respective divisors exhibiting the required progression, so that the equation has no linear factor. If, as another example, we take $\phi(y)=y^5-13y+6=0$, we find $\phi(-1)=18$, $\phi(0)=6$, $\phi(1)=-6$, in which we may discern the divisors 1, 2, 3, causing us to suspect that $\phi(-2)=0$, which is the case. If the first coefficient of $\phi(y)$ is a instead of 1, the common difference must be some factor of a. In searching for a quadratic factor I usually employ the following method, which may be novel,* and which applies to equations of other degrees as well as to the quintic.

4. In the example $y^5-5y-3=0$, we had $\phi(-1)=1$, $\phi(0)=-3$, $\phi(1)=-7$. Continuing, we have $\phi(2)=19$, $\phi(3)=225$, $\phi_4=1001$, $\phi_5=3097$, etc. Avoiding negative products, in order if possible to consider only positive divisors, we take the series as follows:

$$\begin{aligned}
\phi(2) &= 19 \quad = 1\times 19,\\
\phi(3) &= 225 \ = 5\times 45 \ = 9\times 25,\\
\phi(4) &= 1001 = 7\times 143 = 11\times 91 = 13\times 77,\\
\phi(5) &= 3097 = 19\times 163.
\end{aligned}$$

What we have now to do is to look among the respective divisors for an ascending progression: if the differences increase regularly by 2 (more generally, by twice some factor of a), there is a quadratic factor. Such a progression appears in 1, 5, 11, 19. Carrying it back two steps to $\phi(0)$, we have $\phi(1)=-1\times 7$, $\phi(0)=-1\times 3$. The factors of $\phi(y)$ are therefore y^2+py-1, y^3+ry^2+sy+3. It will be observed that the complementary divisors, 3, 7, 19, 45, 91, etc., have as differences the series 4, 12, 26, 46, etc., which have as second differences 8, 14, 20, etc., the constant third difference being 6. In general, any equation of the n^{th} degree having integral coefficients, the first being 1, and having a quadratic factor, must show a series of divisors having 2 as their uniform second difference, the complementary divisors having $(n-2)!$ as their uniform difference, of degree $n-2$. (If the first coefficient is $a=mp$ instead of 1, the uniform differences will be $2m$ and $(n-2)!\,p$ respectively.) We may apply this system, obviously, in seeking for cubic or even larger factors, any factor of degree k yielding a series of which the k^{th} difference is always $k!$, or for the general

*Newton's method of factoring involves eventually only a series having uniform first differences, and to that extent is simpler, but it requires much more preparation.

form $m.k!$, according to an elementary principle of the theory of finite differences.

5. Thus far we have found only the final term of the quadratic or other factor. The theory of finite differences gives us the remaining coefficients of this factor at once. We have in fact found $\psi(0)$, $\psi(1)$, $\psi(2)$, etc., factors of $\phi(0)$, $\phi(1)$, $\phi(2)$, etc., from which, pursuing the same example, we establish the following scheme of differences for the quadratic factor:

y.	$\psi(y)$.	$\Delta\psi(y)$.	$\Delta^2\psi(y)$.
0	-1	0	2
1	-1	2	2
2	1	4	
3	5		

and for the cubic factor:

y.	$\chi(y)$.	$\Delta\chi(y)$.	$\Delta^2\chi(y)$.	$\Delta^3\chi(y)$.
0	3	4	8	6
1	7	12	14	6
2	19	26	20	
3	45	46		
4	91			

A known formula in finite differences is

$$f(y)=f(0)+y\Delta f(0)+\tfrac{1}{2}y(y-1)\Delta^2 f(0)+\frac{1}{2.3}y(y-1)(y-2)\Delta^3 f(0)+\ldots,$$

from which we find at once the quadratic factor of x^5-5x-3 to be $-1+y(y-1)$ and the cubic factor to be $3+4y+4y(y-1)+y(y-1)(y-2)$. The same process may be followed, as suggested in the previous paragraph, in ascertaining factors of still higher degrees, when the given equation is, say, of the n^{th} degree, and the factors are respectively of the k^{th} and $(n-k)^{\text{th}}$.

6. If a quintic is reducible we do not need to inquire, except perhaps when we are looking for a case for purposes of illustration, whether it is also resolvable. If we find it to be irreducible, the construction of a resolvent and its examination for a rational root will determine the question of resolvability. It is known, however (see paragraph 16 farther on), that resolvable quintics have

either five real roots or one real and four imaginary. A quintic known to have just two imaginary roots may, whether reducible or not, be classed at once as unresolvable. Means for recognizing equations having two imaginary roots, often by mere inspection, may be found in a paper read by me before this Society at its summer meeting in 1894, and published in vol. XVII of the American Journal of Mathematics, its title being "A Method for Calculating Simultaneously all the Roots of an Equation." A glance, for example, at such quintics as $x^5+x^3-11x^2-2x+3=0$, $x^5-13x^3-9x+1=0$, $x^5-17x-1=0$, is enough to make sure that they are not resolvable. The following schedule will assist the reader in such cases, it being understood that there must be a marked distinction between the large or "dominant" coefficients and the rest, which must be relatively unimportant; and if such a distinction does not already exist, the equation must be so transformed linearly as to create it. In this schedule the unimportant coefficients are indicated by dots, the coefficient of x^5 by 1, and the other dominant coefficients by D or, when the sign is important, by $+$ or $-$ or $\pm$ or $\mp$.

Arrangements of Coefficients which Indicate Two Imaginary Roots.

x^5	x^4	x^3	x^2	x	1
1	.	$+$	D	D	D
1	.	$+$	$\pm$	.	$\mp$
1	.	$+$	.	$-$	D
1	.	.	$\pm$	.	$\mp$
1	.	.	D	D	D
1	.	.	.	$-$	D
1	$\pm$	.	$\pm$	.	$\mp$
1	$\pm$	.	$\pm$	D	D
1	D	.	.	D	D
1	$\pm$	.	.	.	$\mp$
1	.	$-$	.	$-$	D
1	D	$\pm$	.	$\pm$	D
1	D	D	.	.	D
1	.	$-$	.	.	D
1	D	D	$\pm$	.	$\pm$
1	.	$-$	$\pm$	.	$\pm$
1	$\pm$	.	$\mp$	.	$\mp$

The word "span" is used, in the paper cited, for the space, or difference in the degrees of the exponents, between one dominant and the next. Thus, in the first form given in the foregoing schedule, there is a quadratic span followed by three linear spans. An "unlike span" is bounded by two dominants of unlike signs, and a "like span" by two dominants of like signs. The key to the schedule consists in the observation that two imaginary roots are, in a quintic, indicated always either by a cubic span, a like quadratic, or an unlike quartic.

7. Having noted certain methods for detecting equations which are either reducible or known to be unresolvable because having just two imaginary roots, we shall hereafter confine our attention to resolvable equations. To resolve any resolvable quintic of the form (2) it is sufficient, and when the quintic is irreducible it is necessary, to assume $y = u_1 + u_2 + u_3 + u_4$ and to determine the values of the four u's.* The paper of 1885 already referred to, "Analysis of Quintic Equations," contains not only a new resolvent, but also immediate formulæ expressing the roots of the quintic when a root of the resolvent is known. While these formulæ were derived in a manner not devoid of utility,† the method of proof now to be presented will be found far simpler. The expression for r_2 is also simplified greatly.

8. Let there be four quantities, called elements, namely, u_1, u_2, u_3, u_4, and let certain functions of these elements be defined as follows:

---

* Those critical cases in which one or more of the u's disappear are treated in the earlier paper, paragraphs 7 and 33, and will receive some attention farther on.

† I may be excused for citing the judgment on this point of one or two competent critics. Says Cayley (Collected Mathematical Papers, IV, 612): "McClintock completes in a very elegant manner the determination of the roots of the quintic equation . . . . I reproduce this solution." Cayley's reproduction, or rather paraphrase, occupies nearly five of his quarto pages. The key to every improvement which I made in 1884 and 1885 lay in the recognition of the rational character of the quantities v and s, which for earlier writers were merely squares of quantities which they employed, without regard to their irrationality, as fundamental features of their systems, and to which I attributed distinct symbols; in the discovery and use of the all-important rational quantity t, connected with v and s by the relation $s = t^2 v$, and in the employment (which I recommended urgently by various arguments) of the rational symbols t and v in the discussion of the mechanism of the quintic and in the formulation of the two fundamental equations. It gave me much satisfaction to find these improvements shortly afterwards adopted by Professor Young (American Journal, X, 114), whose y and $-\frac{1}{2}t$ corresponded to my v and t. His formulæ for the roots, p. 114, corresponded with my Nos. 84 and 82, vol. VIII, pp. 67–8.

$$
\begin{aligned}
y &= u_1 + u_2 + u_3 + u_4, && (4)\\
\gamma &= -\tfrac{1}{2}(u_1u_4 + u_2u_3), && (5)\\
v^{\frac{1}{2}} &= \tfrac{1}{2}(u_1u_4 - u_2u_3), && (6)\\
\delta &= -\tfrac{1}{2}(u_2^2u_1 + u_3^2u_4 + u_1^2u_3 + u_4^2u_2), && (7)\\
t &= \tfrac{1}{2}v^{-\frac{1}{2}}(u_2^2u_1 + u_3^2u_4 - u_1^2u_3 - u_4^2u_2), && (8)\\
p &= v^{\frac{1}{2}}(u_2^2u_1 - u_3^2u_4)(u_1^2u_3 - u_4^2u_2), && (9)\\
\varepsilon &= \gamma^2 + 3v - u_1^3u_2 - u_4^3u_3 - u_2^3u_4 - u_3^3u_1, && (10)\\
r_1 &= u_1^5 + u_2^5 + u_3^5 + u_4^5, && (11)\\
r_2 &= u_1^5 + u_4^5 - u_2^5 - u_3^5, && (12)\\
q_1 &= \tfrac{1}{2}(u_1^5 - u_4^5), && (13)\\
q_2 &= \tfrac{1}{2}(u_2^5 - u_3^5), && (14)\\
s_1 &= \tfrac{1}{2}(q_1^2 + q_2^2), && (15)\\
s_2 &= \tfrac{1}{2}(q_1^2 - q_2^2), && (16)\\
\zeta &= -r_1 - 20tv. && (17)
\end{aligned}
$$

The following relations may be proved at once by mere expansion in terms of the elements:

$$
\begin{aligned}
& y^5 + 10\gamma y^3 + 10\delta y^2 + 5\varepsilon y + \zeta = 0, && (18)\\
& p = \gamma\delta^2 - \gamma t^2v + (v - \gamma^2)(\varepsilon - \gamma^2 - 3v), && (19)\\
& p^2 = (\delta^2 - t^2v)^2v + 8(\gamma^2 - v)(\gamma\delta^2 + \gamma t^2v + 2\delta tv)v + 16(\gamma^2 - v)^3v, && (20)\\
& 25tv^2 + (\zeta - \varepsilon t + \delta t^2 - \gamma t^3 - 10\gamma^2 t)v - \gamma t(\gamma\varepsilon - \gamma^3 - \delta^2) - \gamma^2\zeta + 2\gamma\delta\varepsilon - \delta^3 = 0, && (21)\\
& r_2 = [(\zeta + t\varepsilon)\gamma - (\varepsilon + t\delta)\delta + (\delta + t\gamma)(v - \gamma^2) + 12\gamma tv + t^3v]\,v^{-\frac{1}{2}}, && (22)\\
& s_1 = \tfrac{1}{16}(r_1^2 + r_2^2) + \gamma^5 + 10\gamma^3v + 5\gamma v^2, && (23)\\
& s_2 = \tfrac{1}{8}r_1r_2 - (5\gamma^4 + 10\gamma^2v + v^2)v^{\frac{1}{2}}, && (24)
\end{aligned}
$$

$$
\left.
\begin{aligned}
u_1^5 &= \tfrac{1}{4}r_1 + \tfrac{1}{4}r_2 + \sqrt{(s_1 + s_2)},\\
u_2^5 &= \tfrac{1}{4}r_1 - \tfrac{1}{4}r_2 + \sqrt{(s_1 - s_2)},\\
u_3^5 &= \tfrac{1}{4}r_1 - \tfrac{1}{4}r_2 - \sqrt{(s_1 - s_2)},\\
u_4^5 &= \tfrac{1}{4}r_1 + \tfrac{1}{4}r_2 - \sqrt{(s_1 + s_2)}.
\end{aligned}
\right\} \qquad (25)
$$

Strictly speaking, only the first, numbered (18), of these relations requires proof by substitution of the elements of which the symbols are functions. No. (22) may be derived from the value of r_2 stated (paragraphs 3 and 31) in the paper of 1885, viz.

$$
\left.
\begin{aligned}
r_2 = (\gamma^2 - v)^{-1}(&12\gamma tv^2 - \delta v^2 - t^3v^2 + 4\gamma^3tv + 2\gamma^2\delta v\\
&+ \gamma\delta t^2v + \delta^2tv - 2\gamma\varepsilon tv + \delta\varepsilon v - \gamma^4\delta + \gamma^2\delta\varepsilon - \gamma\delta^3)\,v^{-\frac{1}{2}},
\end{aligned}
\right\} \qquad (26)
$$

by subtracting from the second member of (26) multiplied by $(\gamma^2 - v)$ the first member of (21) multiplied by γv^{-1}, and dividing the remainder by $(\gamma^2 - v)$; and (26) was shown to be derived from some of the equations here numbered (5–19). The other relations may likewise be derived, as before, from those equations (5–17) here introduced as definitions, but I make special mention of (22), because it presents a notable simplification of the longest one of the formulæ given in the earlier paper for the final exhibition of the roots of the quintic. By eliminating p from (19) and (20) we have, as before,

$$\left.\begin{aligned} &25v^3 + (-t^4 + 14\gamma t^2 + 16\delta t - 35\gamma^3 - 6\varepsilon)v^2 \\ &\quad + (-2c_0t^3 + 2\gamma\delta^2 + 4\gamma^3\varepsilon + 11\gamma^4 + \varepsilon^2)v - c_0^2 = 0, \end{aligned}\right\} \tag{27}$$

where $c_0 = -\gamma^3 + \gamma\varepsilon - \delta^2$.

9. In (18) we have (2), the shortened form of the general quintic, for the solution of which it is therefore necessary to determine the elements from the element-formulæ here numbered (25), by the extraction of fifth-roots, and for the employment of those formulæ we require to know the values of t and v, a subject to be considered further on. One value of each element is obtained without the intervention of the fifth-roots of unity, and such values will be real if the right-hand members of (25) are real; let the values so obtained be used in (4) for the determination of one root of (18), which let us designate as y_1. Before discussing the other roots of the quintic it is desirable to note the relations which exist between the elements.

10. That it is not necessary to determine more than one of the elements by the extraction of a fifth-root has long been known. It has also been shown by researches in the theory of substitutions that the root of the resolvable quintic may have the form $y_1 = u_1 + z_2u_1^2 + z_3u_1^3 + z_4u_1^4$, where only u_1 involves a fifth-root. According to Schläfli,* $u_2 = u_1^2 . u_4^2u_2(u_1u_4)^{-2}$, $u_3 = u_1^2u_3 . u_1^{-2}$, $u_4 = u_1u_4 . u_1^{-1}$. These expressions are almost exactly what we want, so that, modifying two of them slightly, we have now

$$\left.\begin{aligned} u_2 &= z_2u_1^2 = u_4^2u_2(u_1u_4)^{-2} . u_1^2, \\ u_3 &= z_3u_1^3 = u_1^2u_3 . u_1^{-5} . u_1^3, \\ u_4 &= z_4u_1^4 = (u_1u_4)u_1^{-5} . u_1^4. \end{aligned}\right\} \tag{28}$$

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* Cited 1885 in paragraph 19.

The value of u_1^5 is known, by (25), and those of u_1u_4, $u_4^2u_2$, and $u_1^2u_3$ are obtainable from known quantities by the aid of (5–8), it being observed that $u_1u_2u_3u_4 = \gamma^2 - v$ and that $(u_1^2u_3 - u_4^2u_2)^2 = (u_1^2u_3 + u_4^2u_2)^2 - 4u_1u_2u_3u_4 \,.\, u_1u_4$; that is to say,

$$\left.\begin{aligned} u_1^2u_3 &= -\tfrac{1}{2}(\delta + tv^{\frac{1}{2}}) + \tfrac{1}{2}\sqrt{[(\delta + tv^{\frac{1}{2}})^2 - 4(\gamma^2 - v)(v^{\frac{1}{2}} - \gamma)]}, \\ u_4^2u_2 &= -\tfrac{1}{2}(\delta + tv^{\frac{1}{2}}) - \tfrac{1}{2}\sqrt{[(\delta + tv^{\frac{1}{2}})^2 - 4(\gamma^2 - v)(v^{\frac{1}{2}} - \gamma)]}. \end{aligned}\right\} \quad (29)$$

Let us now turn to the question of ascertaining the other four roots of the quintic, having found $y_1 = u_1 + u_2 + u_3 + u_4 = u_1 + z_2u_1^2 + z_3u_1^3 + z_4u_1^4$.

11. Instead of taking u_1 as the fifth-root of u_1^5, let us take ωu_1, where ω is an imaginary fifth-root of unity, and let us denote the corresponding root of the quintic by y_5. Then $y_5 = \omega u_1 + z_2\omega^2u_1^2 + z_3\omega^3u_1^3 + z_4\omega^4u_1^4 = \omega u_1 + \omega^2u_2 + \omega^3u_3 + \omega^4u_4$. Proceeding in like manner, we derive the following schedule, the remaining fifth-roots of unity being ω^2, ω^3, ω^4, regard being had to the relations $\omega^5 = 1$, $\omega^6 = \omega$, etc.

$$\left.\begin{aligned} y_5 &= \omega u_1 + \omega^2u_2 + \omega^3u_3 + \omega^4u_4, \\ y_4 &= \omega^2u_1 + \omega^4u_2 + \omega u_3 + \omega^3u_4, \\ y_3 &= \omega^3u_1 + \omega u_2 + \omega^4u_3 + \omega^2u_4, \\ y_2 &= \omega^4u_1 + \omega^3u_2 + \omega^2u_3 + \omega u_4, \\ y_1 &= u_1 + u_2 + u_3 + u_4. \end{aligned}\right\} \quad (30)$$

If we multiply the fourth line by ω, the third by ω^2, and so on, and add all five together, recollecting that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$, we derive $5u_1 = y_1 + \omega y_2 + \omega^2y_3 + \omega^3y_4 + \omega^4y_5$. If we proceed similarly with ω^2, ω^4, etc.; then with ω^3, ω^6, etc.; then with ω^4, ω^8, etc., we have finally the well-known definitions of the elements according to the theory of Bezout and Euler, $5u_m = y_1 + \omega^m y_2 + \omega^{2m}y_3 + \omega^{3m}y_4$, where m is 1, 2, 3, or 4. We have thus ended this sketch of a new theory of the quintic by exhibiting the elements as functions of the roots, having begun by defining y as the sum of four elements, in precisely the reverse of the usual order.*

* This reverse method will be found applicable to equations of all degrees. Thus, for the cubic, we may begin by defining $y = u_1 + u_2$, $\gamma = -u_1u_2$, $\delta = -u_1^3 - u_2^3$, and then prove by substitution that $y^3 + 3\gamma y + \delta = 0$; then, as $(u_1^3 - u_2^3)^2 = \delta^2 + 4\gamma^3$, it follows that $u_1^3 = -\frac{1}{2}\delta + \frac{1}{2}\sqrt{(\delta^2 + 4\gamma^3)}$, and $u_2^3 = -\frac{1}{2}\delta - \frac{1}{2}\sqrt{(\delta^2 + 4\gamma^3)}$. Again, for the biquadratic, let $y = u_1 + u_2 + u_3$, $\gamma = -\frac{1}{8}(u_1^2 + u_2^2 + u_3^2)$, $\delta = -2u_1u_2u_3$, $\varepsilon = u_1^4 + u_2^4 + u_3^4 - 2u_1^2u_2^2 - 2u_1^2u_3^2 - 2u_2^2u_3^2$, whence $y^4 + 6\gamma y^2 + 4\delta y + \varepsilon = 0$. Then, since $u_1^2 + u_2^2 + u_3^2 = -8\gamma$, and $u_1^2u_2^2 + u_1^2u_3^2 + u_2^2u_3^2 = \frac{1}{4}(9\gamma^2 - \varepsilon)$, and $u_1^2u_2^2u_3^2 = \frac{1}{4}\delta^2$, we may assign a cubic equation of which the roots shall be u_1^2, u_2^2, and u_3^2, namely, $u^6 + 8\gamma u^4 + \frac{1}{4}(9\gamma^2 - \varepsilon)u^2 - \frac{1}{4}\delta^2 = 0$. While by this method certain relations are assumed in the definitions, it has the advantages of lucidity and succinctness in exhibiting the mechanism of solution.

12. By substituting in any of the definitions (5–17) the values of the elements, just found, in terms of the roots, we shall have the quantities in question exhibited as functions of the roots. In this way we may derive the customary expressions for γ, δ, ε, ζ, but we are now particularly concerned with v and t. Thus,

$$25\,(u_1u_4 - u_2u_3) = 50v^{\frac{1}{2}} = (\omega + \omega^4 - \omega^2 - \omega^3)(y_1y_2 + y_2y_3 + y_3y_4 + y_4y_5 \\ + y_5y_1 - y_1y_3 - y_2y_4 - y_3y_5 - y_4y_1 - y_5y_2). \quad (31)$$

If ϕ represent the latter bracket, we have (since $\omega + \omega^4 - \omega^2 - \omega^3 = \pm\sqrt{5}$) $50v^{\frac{1}{2}} = \pm\sqrt{5}\,.\,\phi$, whence $500v = \phi^2$. Any other system of designating the subscripts of y in (30) will produce in (31) one or other of the six forms in which $\pm\phi$, and therefore v, can be expressed as a function of the roots. The denominator of t, as defined in (8), is $2v^{\frac{1}{2}} = \pm 5^{-2}\sqrt{5}\,.\,\phi$, and its numerator is $u_2^2u_1 + u_3^2u_4 - u_1^2u_3 - u_4^2u_2$, the value of which may similarly be found to be $\pm 5^{-2}\sqrt{5}\,.\,(y_1y_2y_5 + y_2y_3y_1 + y_3y_4y_2 + y_4y_5y_3 + y_5y_1y_4 - y_1y_3y_5 - y_2y_4y_1 - y_3y_5y_2 - y_4y_1y_3 - y_5y_2y_4)$ or say $\pm 5^{-2}\sqrt{5}\,.\,\sigma$, so that $t = \sigma\phi^{-1}$. The latter operation is however somewhat intricate and requires special consideration.

13. Let the letter C represent the sum of five similar functions of the roots, comprising a cycle, each function being formed from the one preceding by advancing the subscript of each root involved, y_1 becoming y_2, y_2 becoming y_3, ... y_5 becoming y_1. Thus $\phi = \mathrm{C}y_1y_2 - \mathrm{C}y_1y_3 = \mathrm{C}y_1(y_2 - y_3)$, and $\sigma = \mathrm{C}y_1y_2y_5 - \mathrm{C}y_1y_3y_5 = \mathrm{C}y_1(y_2 - y_3)\,y_5$. As already stated, the substitution in (6) of the values of the elements in terms of the roots produces $50v^{\frac{1}{2}} = \pm\sqrt{5}\,.\,\phi$; but a similar substitution in the numerator of t in (8) does not produce at once $\pm 5^{-2}\sqrt{5}\,.\,\sigma$, but $\pm 5^{-3}\sqrt{5}\,.\,(4\sigma + \xi)$, where $\xi = \mathrm{C}y_1^2(y_3 + y_4 - y_2 - y_5)$, and it is necessary to show that $\xi = \sigma$. Since $y_1 + y_2 + y_3 + y_4 + y_5 = 0$, we have $\xi = \mathrm{C}y_1(y_2 + y_5 + y_3 + y_4)(y_2 + y_5 - y_3 - y_4) = \mathrm{C}y_1(y_2^2 + y_5^2 - y_3^2 - y_4^2 + 2y_2y_5 - 2y_3y_4) = 2\sigma - \xi$, whence $\xi = \sigma$. For, in ξ, $\mathrm{C}y_1^2y_3 = \mathrm{C}y_1y_4^2$, $\mathrm{C}y_1^2y_4 = \mathrm{C}y_1y_3^2$, $\mathrm{C}y_1^2y_2 = \mathrm{C}y_1y_5^2$, and $\mathrm{C}y_1^2y_5 = \mathrm{C}y_1y_2^2$; and, in σ, $\mathrm{C}y_1y_2y_5 = \mathrm{C}y_1y_3y_4$.—Or, we may prove that $\xi - \sigma = 0$ by showing that $\xi - \sigma$ is exactly divisible by $y_1 + y_2 + y_3 + y_4 + y_5 = 0$.

14. For ascertaining the values of t and v, which is all that is necessary in order to exhibit the elements of the roots as in (25), I have nothing to add to

my earlier discoveries, contained in paragraphs 3, 4, 5, 25, 41, 42, of the paper of 1885, which will be summarized in this paragraph for the convenience of the reader. Let there be an auxiliary quintic, $\mathrm{A}_y = t^5 + 10\gamma t^3 + 10\delta t^2 + 5\varepsilon t + \zeta$, with its canonizant, $\mathrm{C}_y = c_0 t^3 + c_1 t^2 + c_2 t + c_3$, and its simplest linear covariant, $\mathrm{L}_y = l_0 t + l_1$, where

$$\left.\begin{aligned} c_0 &= -\gamma^3 + \gamma\varepsilon - \delta^2, \\ c_1 &= -\gamma^2\delta + \gamma\zeta - \delta\varepsilon, \\ c_2 &= -\gamma\delta^2 + \gamma^2\varepsilon + \delta\zeta - \varepsilon^2, \\ c_3 &= \quad 2\gamma\delta\varepsilon - \gamma^2\zeta - \delta^3, \end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned} l_0 &= -15\gamma^4\varepsilon + 10\gamma^3\delta^2 - 2\gamma^2\delta\zeta + 14\gamma^2\varepsilon^2 - 22\gamma\delta^2\varepsilon + \gamma\zeta^2 + 9\delta^4 - 2\delta\varepsilon\zeta + \varepsilon^3, \\ l_1 &= \quad 9\gamma^4\zeta - 20\gamma^3\delta\varepsilon + 10\gamma^2\delta^3 + 8\gamma^2\varepsilon\zeta - 12\gamma\delta\varepsilon^2 - 2\gamma\delta^2\zeta + 6\delta^3\varepsilon + \delta\zeta^2 - \varepsilon^2\zeta. \end{aligned}\right\} \tag{33}$$

Let a value of t be found by the numerical solution of my resolvent,

$$\mathrm{A}_y\mathrm{L}_y - 25\mathrm{C}_y^2 = 0, \tag{34}$$

which may be written thus:

$$\left.\begin{aligned} &(l_0 - 25c_0^2)\,t^6 + (l_1 - 50c_0c_1)\,t^5 + 5\,(2\gamma l_0 - 5c_1^2 - 10c_0c_2)\,t^4 \\ &+ 10\,(\gamma l_1 + \delta l_0 - 5c_0c_3 - 5c_1c_2)\,t^3 + 5\,(2\delta l_1 + \varepsilon l_0 - 5c_2^2 - 10c_1c_3)\,t^2 \\ &+ (5\varepsilon l_1 + \zeta l_0 - 50c_2c_3)\,t + \zeta l_1 - 25c_3^2 = 0. \end{aligned}\right\} \tag{35}$$

Also, after t is known, let v be found by using either of these expressions:

$$v = -\mathrm{C}_y\mathrm{A}_y^{-1} = -\tfrac{1}{25}\,\mathrm{L}_y\mathrm{C}_y^{-1}. \tag{36}$$

For the broader form (1) of the general quintic, namely, $ax^5 + 5bx^4 + \ldots. = 0$, where $x = y - ba^{-1}$, let there be an auxiliary quintic in $\tau = t - ba^{-1}$, namely $\mathrm{A} = a\tau^5 + 5b\tau^4 + \ldots.$, with its canonizant C and its simplest linear covariant L, and we may obtain τ by means of my broader resolvent, of which (34) is a special case,

$$\mathrm{AL} - 25\mathrm{C}^2 = 0, \tag{37}$$

and v from

$$v = -\mathrm{CA}^{-1}a^{-2} = -\tfrac{1}{25}\,\mathrm{LC}^{-1}a^{-2}. \tag{38}$$

It will be found that ϕ is the same function of x as of y, but this is not true of σ, which is equivalent to $t\phi$. Let $\tau\phi = \psi$; then $\psi = t\phi - ba^{-1}\phi = \sigma - ba^{-1}\phi$, and $\tau = \psi\phi^{-1}$, where ψ is the same function of the roots of (1) as σ is of the roots of (2). As τ is a function of the roots which is the reciprocal of the same function of the reciprocals of the roots, it is a covariant function, and having six values is to be determined by a covariant sextic (37), equivalent to $(\phi_1\tau - \psi_1)(\phi_2\tau - \psi_2)\ldots.(\phi_6\tau - \psi_6) = 0$.

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$C_y A_y^{-1} = \frac{1}{2}$. Then,

$l^2 = 1$, whence if $l = 1$ we get $q = -\frac{1}{2}$ and $r = -1$; and that $l = 1$, and not -1, is corroborated by (39). In this way the original parameters are rediscovered. Or, more directly, we may find the parameters of resolvable quintics in general, including those of the quintic just taken as an illustration, by putting

$$l = \frac{\delta^2 + t^2 v + 4\gamma(\gamma^2 - v)}{2\delta t - 4(\gamma^2 - v)}, \tag{42}$$

deriving now the values of q, r, w, from (40). That the two expressions for l, (39) and (42), are consistent will be seen upon substituting for the symbols γ, δ, v, in (42) their values in terms of q, r, s, t, l, as given in (40). If we define l as in (42), $q = \gamma l^{-1}$, $r = \delta l^{-1} - t$, and w as determined by (40, ε), it remains only to prove that $v = l^2(1 + w^2)^{-1}$ in order to establish all the statements of this paragraph.

18. That $l^2 = v(1 + w^2)$ may be proved either algebraically, employing only the rational symbols γ, δ, t, p, v; or by defining w and l in terms of the elements. Let w be such that $p = wv[2\delta t - 4(\gamma^2 - v)]$. This substituted in (19) produces (40, ε). This expression for p, again, substituted in (20), gives $w^2v^2[2\delta t - 4(\gamma^2 - v)]^2 =$ second member; and the addition to each side of $v^2[2\delta t - 4(\gamma^2 - v)]^2$ transforms the second member into a perfect square multiplied by v, that is to say,

$$v^2(1 + w^2)[2\delta t - 4(\gamma^2 - v)]^2 = v[\delta^2 + t^2 v + 4\gamma(\gamma^2 - v)]^2. \tag{43}$$

Hence, if l be defined as in (42), $v(1 + w^2) = l^2$. Let us now consider the other mode of proof, by which w and l are defined as functions of the elements.

19. Let $m = [u_1^2 u_3 - u_4^2 u_2] \div [u_2^2 u_1 - u_3^2 u_4]$, and let

$$\left.\begin{aligned} w &= \frac{2m}{m^2 - 1}, \\ l &= v^{\frac{1}{2}} \frac{m^2 + 1}{m^2 - 1}. \end{aligned}\right\} \tag{44}$$

Then $1 + w^2 = \left(\dfrac{m^2 + 1}{m^2 - 1}\right)^2$, and $l^2 = v(1 + w^2)$. That

$$w = \frac{\gamma(\delta^2 - t^2 v) + (\gamma^2 - v)(\gamma^2 + 3v - \varepsilon)}{v[2\delta t - 4(\gamma^2 - v)]}, \tag{45}$$

and

$$l = \frac{\delta^2 + t^2 v + 4\gamma(\gamma^2 - v)}{2\delta t - 4(\gamma^2 - v)}, \tag{42}$$

may be verified by substituting for all the symbols involved their values in terms of the elements. Therefore w and l are rational when t is rational, though m is usually irrational. Then q and r must be defined, as in (40), in terms of γ, δ, t, l, that is to say, $q = \gamma l^{-1}$, $r = \delta l^{-1} - t$. A simple illustration is supplied by $u_1 = 1$, $u_2 = -1$, $u_3 = \frac{1}{2}$, $u_4 = 2$, $\gamma = -\frac{3}{4}$, $\delta = 1$, $\varepsilon = \frac{33}{8}$, $m = 9$, $w = \frac{9}{40}$, $l = \frac{41}{32}$, $v = \frac{25}{16}$, $t = 2$. Since $5u = y_1 + \omega^n y_2 + \omega^{2n} y_3 + \omega^{3n} y_4$, where n is 1, 2, 3, or 4, and ω is a fifth-root of unity, as in paragraph 11, it becomes a mere matter of substitution to express w and l, and therefore q and r, as rational functions of the roots of the quintic. The rational parameter t is a rational function of the roots and also of the elements, and is obtained from the coefficients by means of the resolvent; the other three rational parameters are rational functions of the roots, and also of the elements, and also of the coefficients and t; while the coefficients are, as here assigned, rational functions of the four rational parameters, as well as of the usually irrational roots and of the usually irrational elements. For the fuller form (1) of the quintic a fifth rational parameter is obviously to be assigned, namely, the quantity ba^{-1}.

20. Having now at our disposal the skeleton, so to speak, of the general resolvable quintic, we can adjust the parameters so as to produce at will resolvable quintics having specific forms or properties. When $q = 0$, $\gamma = 0$; when $r = -t$, $\delta = 0$; when m is rational, $1 + w^2 = \left(\frac{m^2+1}{m^2-1}\right)^2$ because $w = \frac{2m}{m^2-1}$, so that v is the square of a rational quantity, and so on. If, for example, we give to w a rational value such that $w^2 = 5n^2 - 1$, where n is rational, we shall have $\frac{1}{5} l^2 n^{-2} = v = \frac{1}{500} \phi^2$, whence $\phi = 10 l n^{-1}$; and in this case ϕ has a rational value, and the historic resolvent of Jacobi and Cayley becomes available, as having a rational root. In order, therefore, to construct a quintic resolvable by ϕ, we must take rational values for w and n such that $w^2 = 5n^2 - 1$, the other three parameters, q, r, t, remaining at our arbitrary disposal. How small a proportion of all resolvable quintics are resolvable by ϕ may be appreciated when we reflect that, taking for the moment integral values only, every such value given to w produces a resolvable quintic, while of the numbers up to 100 only 2 and 38 appear to be admissible for a value of w consistent with the resolvent in ϕ.—In certain critical cases, which we shall now examine, the system of construction presented in (39) and (40) requires modification.

: $\phi = 0$, $v = 0$, $t = \sigma\phi^{-1} = \infty$, $q = q_a t$, $r = r_a t$, $l_a = lt$
$u_4 = u_2 u_3$. To construct a resolvable quintic havin
disposable three rational parameters, q_a, r_a, w, an
we obtain the value of l_a (see 39) from

$$t^3 w^3 - r_a^2 t^3 (1 + w^2)] \div 4 q_a t q_a^2 t^2 (1 + w^2)$$
$$w^2 - r_a^2 (1 + w^2)] \div 4 q_a^3 (1 + w^2); \qquad (46$$

: $l_a t^{-1} = 0$, we have $v = 0$, $\gamma = q_a l_a$, $\delta = (r_a + 1) l_a$
ere $s = t^2 v = l_a^2 \div (1 + w^2)$. For determining ζ w
multiplying it throughout by $(\gamma^2 - v)^2$ and by subst
ivalent of ε as given by (40), whence we derive thi
available for all values of t and v:

$$- 2\gamma t v + \delta v)(\delta^2 - t^2 v)$$
$$\delta v + 8\gamma^2 t v) + (\gamma^2 t + tv - 2\gamma\delta)(2\delta t - 4\gamma^2 + 4v) vw. \quad (47$$

nsideration, $v = 0$, $tv = 0$, $t^2 v = s$, and (47) becomes
by γ^4,

$$= 2\gamma\delta + \gamma^{-2}\delta(\delta^2 - s + 2sw). \qquad (48$$

gn $q_a = -\frac{8}{13}$, $r_a = \frac{8}{13}$, $w = \frac{5}{12}$, we have, by (46)
, $\delta = -8$, $s = 36$, $\varepsilon = 29$, and, by (48), $\zeta = -480$
for the special case $v = t^{-1} = 0$, the solvable equatio
$-480 = 0$. This equation served, in the 1885 paper
thod then presented for solving equations for whic
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ish, while the other coefficients do not. Conversely
sort has presented itself, and has been solved as to th
ine the three parameters by the aid of (42), which i

$$l_a = lt = (\delta^2 + s + 4\gamma^3) \div 2\delta, \qquad (49$$

and r_a, and of (48), which supplies the value of w.*

3: $\phi = 0$, $\sigma = 0$, t finite but indeterminate $= \sigma\phi^-$
this case of no account whatever, we may allow it t

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quintics, as published in the paper of 1885, consists in the formul
$-\frac{1}{4}(4\gamma s - \gamma^4 - \delta\zeta + \varepsilon^2)$, $s_1 = \frac{1}{18}(r_1^2 + r_2^2) + \gamma^5$, $s_2 = \frac{1}{3} r_1 r_2$, these valu
sent paper.

vanish, knowing that similar results must follow, whatever value be assigned to t. In this case B we have, from (40) and (41),

$$\left.\begin{aligned} \varepsilon &= \gamma^2 + \gamma^{-1}\delta^2, \\ \zeta &= \gamma\delta + \gamma^{-1}\delta\varepsilon. \end{aligned}\right\} \tag{50}$$

These novel expressions will enable us both to construct quintics of this class by assigning values to γ and δ, and to recognize such quintics, whenever they may present themselves, among those in which $v = 0$. In this remarkable case, not only does the first coefficient of the resolvent vanish, as in case A, but all the other coefficients likewise vanish identically. As regards the affiliation of these quintics to the general scheme of (39) and (40), I am disposed to divide them into two classes, those namely in which w is infinite, and those in which w is indeterminate. The latter class will be considered separately. As for the former, dropping t and putting $w^{-1} = 0$, we have from (39),

$$l = -r^2 \div 4(q^3 + q^2). \tag{51}$$

Since $\gamma = ql$ and $\delta = rl$, it may appear that this mode of construction is less general, as it certainly is less simple, than by merely assigning values to γ and δ; but this is not the fact, as will be seen on referring to (42), which becomes

$$l = -\gamma - \tfrac{1}{4}\delta^2\gamma^{-2}, \tag{52}$$

an expression which enables us to find rational values of q and r for any assigned values of γ and δ. Since $w^{-1} = 0$, we have, from (44), $m^2 - 1 = 0$, $m = \pm 1$; that is to say,

$$u_1^2u_3 - u_4^2u_2 = \pm(u_2^2u_1 - u_3^2u_4). \tag{53}$$

Since $v = 0$ and $tv^{\frac{1}{5}} = 0$, we have, from (6) and (8),

$$\begin{aligned} u_1u_4 &= u_2u_3, \\ u_2^2u_1 + u_3^2u_4 &= u_1^2u_3 + u_4^2u_2. \end{aligned}$$

From these, if we take the upper sign in (53), we find that u_1, u_2, u_3, u_4, are in geometrical progression, say $u_2 = ku_1$, $u_3 = k^2u_1$, $u_4 = k^3u_1$; or, if we take the lower sign, we reach a similar progression, $u_1 = ku_3$, $u_4 = k^2u_3$, $u_2 = k^3u_3$. Whenever, therefore, the elements form a geometrical progression, we have a quintic of this sort, and *vice versa*. Let us, for example, take $k = -2^{\frac{1}{5}}$, $u_1 = 2^{\frac{1}{5}}$, $u_2 = -2^{\frac{2}{5}}$, $u_3 = 2^{\frac{3}{5}}$, $u_4 = -2^{\frac{4}{5}}$: from (6), (8), (44), we have $v = 0$, $tv^{\frac{1}{5}} = 0$, $\phi = 0$, $\sigma = 0$, $s = 0$, $w^{-1} = 0$; from (5), (7), we have $\gamma = 2$, $\delta = 2$; and from (50), $\varepsilon = 6$, $\zeta = 10$. If these values be substituted in the resolvent (34), every term

vanishes. Conversely, the values of k and u_1 can be derived readil and (7), which give $\gamma = -k^3 u_1^2$, $\delta = -(1+k^5)k^2 u_1^3$. The quintic o may thus be solved in a manner differing from the special solution giv graph 8 of the paper of 1885,* though not more simple.

23. CRITICAL CASE C: Same as case B, with the added restr $4\gamma^3 + \delta^2 = 0$, so that $\varepsilon = -3\gamma^2$, $\zeta = -2\gamma\delta$. In this case we have preceding paragraph, $4\gamma^3 = -4k^9 u_1^6$, and $\delta^2 = (k^4 + 2k^9 + k^{14}) u_1^6$, the su whence $1 - k^5 = 0$, and k must be either 1 or ω, a fifth-root of unit pondingly, u_1 must be either g or $g\omega$, where g is rational. In th referring to (30), we have $y_1 = 4u_1$; in the other case, writing ωg for we have $y_1 = \omega g + \omega^2 g + \omega^3 g + \omega^4 g$, and $y_2 = 4g$. In each case there $4g$, and four other equal roots, each $-g$. The form of these quinti fore $y^5 - 10g^2y^3 - 20g^3y^2 - 15g^4y - 4g^5 = 0$. This informs us that ments are to be identical they may have a rational value assigned to that any quintic of this form must have identical elements. For the are also comprised under case B, $v = 0$, $tv^{\frac{1}{2}} = 0$, and every term of th vanishes. My only reason for discussing them apart from case B i appear to require distinct consideration from the point of view of th tive formulæ (39) and (40). If we refer to (44), we see that wh $= u_3 = u_4$, or when $u_2 = \omega u_1$, $u_3 = \omega^2 u_1$, $u_4 = \omega^3 u_1$, or in short when becomes indeterminate, and thus w becomes indeterminate, instead as observed under case B. In case C we have thus both t and w inde while $v = 0$, and therefore $l = 0$. Referring to (39), we find that to case under the general scheme we need to put $r = r_b q$, and $q =$ $l_b = lq$, (39) becomes

$$l_b = -\tfrac{1}{4} r_b^2,$$

and we have $l = 0$, $v = 0$, $\gamma = l_b$, $\delta = r_b l_b$. Since $\gamma = -u_1^2$, it fo $r_b = 2u_1$. It is to be inferred that in case B l has a value other than which has l in its numerator, vanishes because w, an infinite quant tained in its denominator; and that in case C, as may be seen by (52), l vanishes identically, while w becomes indeterminate. The co formula (51), which applies to all other quintics of class B when

* Namely, for use in (25), $r_1 = -\zeta$, $r_2 = -(\gamma^{-2}\delta^3 + 4\gamma\delta)$, $s_1 = \frac{1}{16}(r_1^2 + r_2^2)$, $s_2 = \frac{1}{8} r_1$ be improved by writing $r_2 = -\zeta - 2\gamma\delta$.

not vanish, needs for this class to be replaced by the different constructive formula (54).

24. Critical Case D: $\gamma^3 - v = n^3t^3 = \frac{1}{2}\delta t$, where $n = (\gamma t + \delta)t^{-2}$. The special cases thus far discussed have been those in which $v = 0$, and each of them has in some sort been brought under the general constructive formulæ (39) and (40). Those which we have still to consider are such as cannot properly be included under those formulæ. Our object in framing those formulæ was to solve the diophantine problem presented by the fundamental equations (19–21), in which we have to assign admissible rational values to p, t, and v. To solve this problem in its general form we have found it necessary to make use of the relationships embodied in (42) and (43). In the special case now considered, we find that both sides of (43), and both numerator and denominator of the value of l shown in (42), all vanish identically: that is to say,

$$\left.\begin{aligned} \delta^3 + t^3v + 4\gamma(\gamma^3 - v) &= 0, \\ 2\delta t - 4(\gamma^3 - v) &= 0. \end{aligned}\right\} \tag{55}$$

Since these equations impose two restrictions upon the values to be assigned to γ, δ, t, v, we have only two parameters at our disposal, t and n. To produce a resolvable quintic of this sort, we must therefore assign rational values to t and n, after which we have, by successive substitution in (40) and (41),

$$\left.\begin{aligned} \delta &= 2n^2t, \\ \gamma &= nt - 2n^2, \\ v &= 4n^3(n - t), \\ \varepsilon &= (5t - 4n)(n^2t - 4n^3), \\ \zeta &= (5t - 4n)^2 . 4n^3. \end{aligned}\right\} \tag{56}$$

For example, let $t = -2$, $n = -\frac{1}{2}$; then $\gamma = \frac{1}{2}$, $\delta = -1$, $\varepsilon = 0$, $\zeta = -32$, and the quintic is $y^5 + 5y^3 - 10y^2 - 32 = 0$. By substitution in (23) and (24) we find $s_1 = s_2 = 0$, and from this we find, by (25), $u_1^5 = u_4^5$, $u_2^5 = u_3^5$. This may mean that $u_1 = \omega u_4$, where ω is a fifth-root of unity, but the only arrangements of such fifth-roots compatible with (5–8) result in exhibiting one root of the quintic as a sum of real elements, so that we may say at once that $u_1 = u_4$, $u_2 = u_3$, and these relations will supply the simplest definition of this special class. All the other special cases have been noticed heretofore, and solutions of them were given in the paper of 1885. This case might be solved by (25), but

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we shall have $l = \frac{1}{4}$, and from (40) and (41) we derive $\gamma = 0$, $\delta = \frac{1}{5}$, $\varepsilon = 0$, $\zeta = \frac{1}{5}$, so that the quintic is

$$y^5 + 2y^2 + \tfrac{1}{5} = 0. \tag{61}$$

To produce this example of a resolvable quintic of the form $y^5 + 10\gamma y^3 + 10\delta y^2 + 5\varepsilon y + \zeta = 0$, in which $\varepsilon = 0$, I have been compelled to reduce two of the four parameters to zero. Whether any method of constructing such quintics can be devised which shall place three parameters at our disposal is a question which the future must determine. When we desire to suppress γ or δ, we have only to put $q = 0$ or $r = -t$; to suppress ε and still have three parameters free is a desideratum, which may eventually prove to be an impossibility.

27. The reader will naturally be interested in the inquiry whether the four parameters are presented in their simplest form. It is obvious that any four quantities which are rational functions of these parameters, and of which the parameters are rational functions, could be made to serve as parameters in lieu of q, r, t, and w. To examine this question let us set before us the conditions which have to be satisfied, namely, from (42–44),

$$l = \frac{\delta^2 + t^2 v + 4\gamma(\gamma^2 - v)}{2\delta t - 4(\gamma^2 - v)} = v^{\frac{1}{2}} \frac{m^2 + 1}{m^2 - 1}, \tag{62}$$

$$l^2 = v(1 + w^2), \tag{63}$$

$$w = \frac{2m}{m^2 - 1}, \tag{64}$$

$$m = \frac{u_1^2 u_3 - u_4^2 u_2}{u_2^2 u_1 - u_3^2 u_4}. \tag{65}$$

Considered as functions of the roots or of the elements, t is of weight 1, γ and l of weight 2, δ of weight 3, v of weight 4, w and m having no weight. Our object is to select as parameters four quantities which shall be rational functions of t, γ, δ, v, and w, and of which the latter shall be rational functions. It is necessary to include w in this list, or at any rate w multiplied by δ, t, or v, or by any two, or by all three, since the expression for ε in (40) includes $\delta t v w$. It would not improve (63) were we to substitute for w its equivalent in terms of weighted quantities such as δw or tw, nor could any suitable function of w, taking its place, be represented so simply as a function of the elements as w in (64). It is not conceivable that the form in which w is presented can be improved upon, though it might of course be affected by a numerical multiplier.

It will not be forgotten that m is not admissible as a parameter, being usually irrational. The more we examine t the more we shall be convinced that, as representing weight 1, it cannot be replaced as a parameter by anything else than nt, where n is some number, positive or negative, a change which appears to present no advantage. We have therefore to find two parameters which, with t and w, shall take the place of γ, δ, v, and l, in (62) and (63). The first temptation is to assume that the weights of γ, δ, v, and l, are derivable from t, as in $\gamma = ht^2$, $\delta = it^3$, $l = jt^2$, but the only good purpose served by this suggestion is to make it clear that γ, δ, and l must be made simple functions of some quantity such that the employment of (62) shall exhibit such quantity as a rational function of t, w, and two other parameters. As γ and l are of lower weight than δ, it seems desirable to take as such quantity either γ or l. If we assume $l = h\gamma$, and $\delta = (k + ht)\gamma$, we shall derive from (62), upon due substitution,

$$\gamma = \frac{h^2 t^2 w^2 - k^2 (1 + w^2)}{4(h+1)(1 + w^2 - h^2)}, \tag{66}$$

and since $\delta = \gamma(k + ht)$ and $v = h^2\gamma^2 \div (1 + w^2)$, we have here a solution of the difficulty. This is, in fact, the same as (39), provided we write h^{-1} for q and $h^{-1}k$ for r, and multiply both sides of (39) by h^{-1} or q. The use of (66), however, is not to be preferred, since of itself it is rather less simple than (39), while the expressions for δ and v are decidedly less satisfactory. With (66) we cannot even make $\gamma = 0$ without first putting $h = \infty$. We therefore find it necessary to have $\gamma = ql$; and it only remains to consider why we should have $\delta = (r + t)\,l$ instead of, say, $\delta = gl$ or $\delta = (g + 1)\,tl$. The latter is not admissible because in fact δ often vanishes while t does not, and *vice versa.* The reason why it is preferable to write $\delta = (r + t)\,l$, where r is a parameter of weight 1, is that by this arrangement we are enabled to reduce to its simplest form the construction of sets of conjugate resolvable quintics.

28. Quintics may be called conjugate when they have the same resolvent t and the same value of v corresponding to t. It appears that they must also have the same value of γ. The existence of one quintic conjugate to any given resolvable quintic was discovered and pointed out by me some years ago, when I showed how its coefficients could be determined from those of the given quintic, knowing t and v, by the aid of a quadratic equation. The method presented in

the present paper for constructing at will resolvable quintics by the aid of four rational parameters enables us also to construct their conjugates at once. The second quintic, that is to say, the conjugate formerly discovered, may now be constructed by simply changing the sign of the parameter w. A third may be constructed by changing the sign of the parameter r; and a fourth by changing the signs of both w and r. Since v contains w and r only as w^2 and r^2, such changes of sign leave the value of v unaltered. Thus, for example, if we have $t=4$, $q=0$, $r=\pm 2$, $w=\pm 1$, we derive from (39) and (40) these four conjugate quintics, for each of which $t=4$ and $v=2$:

$$\begin{aligned}
y^5 - 120y^2 - 410y - 1200 &= 0,\\
y^5 - 120y^2 + 470y - \ \ 496 &= 0,\\
y^5 - \ \ 40y^2 - \ \ 90y - \ \ 240 &= 0,\\
y^5 - \ \ 40y^2 + 150y - \ \ \ \ 48 &= 0.
\end{aligned}$$

In the special case E, wherein $v=\gamma^2$, and the three parameters are γ, δ, and t, there is no limit to the possible number of conjugates having identical values for t and v. In case D there appear to be no conjugates, and in cases B and C, in which t is indeterminate, the question does not arise. In case A, wherein $v=0$ and $t=\infty$, quintics having the same value for s may be regarded as conjugate. If we attempt to construct a general resolvable quintic which shall have no conjugate, by putting $r=0$ and $w=0$ in (39), we merely get $l=0$, whence $\gamma=0$, $\delta=0$, $\varepsilon=0$.

29. The resolvent (34), and therefore all other possible resolvents, can be expressed rationally in terms of the four parameters, and when so expressed becomes identically zero. As a simple illustration, let us take $q=0$, $r=-t$, so that $\gamma=0$, $\delta=0$, and the quintic (2) is reduced to the trinomial form

$$y^5 + 5\varepsilon y + \zeta = 0. \tag{67}$$

In this case my general resolvent becomes

$$\varepsilon t^6 - \zeta t^5 - 20\varepsilon^2 t^2 - 4\varepsilon\zeta t - \zeta^2 = 0, \tag{68}$$

which will doubtless be found the simplest resolvent possible for this trinomial form. By (40) and (47),

$$\left.\begin{aligned}
\varepsilon &= v\,(4w+3),\\
\zeta &= tv\,(4w-22),
\end{aligned}\right\} \tag{69}$$

where, by (40), $v = l^2 \div (1 + w^2)$, and, by (39), $l = \frac{1}{4} t^2$. Substituting for ε and ζ in (68) their values from (69), and dividing throughout by $t^2 v$, we have

$$t^4 (4w + 3) - t^4 (4w - 22) - 20v (4w + 3)^2 \\ - 4v (4w + 3)(4w - 22) - v (4w - 22)^2 = 0, \quad (70)$$

or

$$25t^4 - 400v (1 + w^2) = 0,$$

which is an identity. Students of the quintic who prefer to deal only with the trinomial form (67) will find that the derivation of (68) from (70) by the aid of (69) covers their ground pretty well. Those, however, who suppose that the reduction of the general quintic to the form (67), by means of the Bring-Jerrard transformation, is a physically available process will do well to make a personal experiment by performing that transformation in some numerical case. It will be found far more difficult than the use of the resolvent (34, 37).

30. The expressions (69) just noted for the construction of resolvable quintics of the form $y^5 + 5\varepsilon y + \zeta = 0$ are, though special cases of (39) and (40), equivalent in substance to certain formulæ devised by Professor G. P. Young, which were published, in different forms, in the same number of the American Journal of Mathematics (VII, 170, 178) by that writer under his own name and that of Mr. J. C. Glashan. As I have stated that the use of the quantity t was original with myself, it must be explained that Professor Young employed on this occasion a symbol equivalent to t confined to the trinomial only, a symbol moreover which he introduced late in the discussion to represent the fourth-root of a fraction; and that Mr. Glashan's $-2k$, which happened to correspond to t for this trinomial, had a widely different meaning in its general definition, as we shall see when we come to consider Mr. Glashan's work further on. The same remark applies to his m, which for this trinomial is equivalent to $-w$. I had previously made extensive use of t, as here defined for the general quintic, in vol. VI of the same journal, except that it was there taken with the opposite sign. Concerning the trinomial form in question Mr. Glashan said that "the solvable quintic assumes the form

$$x^5 + 5\left(\frac{3 - 4m}{1 + m^2}\right) k^4 x + 4\left(\frac{11 + 2m}{1 + m^2}\right) k^5 = 0, \quad (71)$$

a form communicated to the present writer [Glashan] by Professor G. P. Young of Toronto University in May, 1883." I dwell with some particularity upon the

history of the formulæ available for constructing resolvable trinomials of the form $y^5 + 5\varepsilon y + \zeta = 0$, because such formulæ are the only ones known hitherto by which resolvable quintics can actually be constructed, otherwise than with rational roots or elements. The two papers mentioned appeared late in 1884 or early in 1885. Later in the same year, 1885, the following formula for the resolvable trinomial was published in the Acta Mathematica* by Runge:

$$x^5 + \frac{5\mu^4(4\lambda + 3)}{\lambda^2 + 1}x + \frac{4\mu^5(2\lambda + 1)(4\lambda + 3)}{\lambda^2 + 1} = 0. \tag{72}$$

Here $\mu = -\frac{1}{2}t$, and $\lambda = \dfrac{4 - 3w}{4w + 3}$. In 1890 the following was published at Moscow† by Bugaieff and Lachtine:

$$(\lambda x)^5 + \frac{(\mu - 1)(\mu - 11)}{4(\mu^2 + 4)}(\lambda x) + \frac{\mu - 11}{2(\mu^2 + 4)} = 0. \tag{73}$$

Here $\lambda = t^{-1}$, and $\mu = 2\dfrac{2 + 11w}{2w - 11}$.

31. With a single exception, the situation up to the present time‡ is this, that except for the trinomial case $y^5 + 5\varepsilon y + \zeta = 0$, no method for constructing resolvable quintics has ever been attempted, apart, of course, from the giving of rational values to the elements or to the roots. The exception consists in certain formulæ, or fragment of a paper, which appeared in 1884 or 1885 under the name of Mr. Glashan as already mentioned. I learn from Mr. Glashan that early in 1883 Professor Young deposited with him the trinomial formula (71) under seal, stating that there was such a formula enclosed; that he thereupon endeavored to produce a corresponding formula to compare with Professor Young's when it should be opened, and communicated his results, without proof, to Professor Young, who subsequently caused his somewhat hasty sketch to be published in the American Journal without his knowledge. Meanwhile, his intricate preliminary work had been mislaid or destroyed. It is thus explained how the paper originated, how it came to appear without demonstration while yet including errors of detail, and why the author made no subsequent correc-

* Cited in the Fortschritte as in vol. VIII; in Weber's Algebra as in vol. VII. The trinomial form given by Weber, I, 626, is erroneous.

† Cited in Fortschritte, XXII, 114.

‡ Perhaps I should mention that I communicated the construction-method of (39) and (40) to Professor E. H. Moore in September, 1894, by letter.

tion. Considering the state of the theory at the time when this remarkable fragment appeared, it must, notwithstanding its imperfections, be recognized as a bold and able advance upon untrodden ground. Had Mr. Glashan had an opportunity of correcting his formulæ before they were published and of supplying the demonstration, they would, although not meant to cover the whole field of resolvable quintics, but expressly introduced as applying only to a group of such quintics,* have constituted a notable advance upon the only formulæ of the sort known up to the present time, those namely for constructing the trinomial $y^5 + 5\varepsilon y + \zeta = 0$. In particular, Mr. Glashan's formulæ, if they had been corrected, would have covered, among other resolvable quintics, the whole of the class of quadrinomials of the form $y^5 + 10\gamma y^3 + 5\varepsilon y + \zeta = 0$, a form which includes as special cases both the trinomial already mentioned and the trinomial $y^5 + 10\gamma y^3 + \zeta = 0$. It has been a matter of some difficulty for me to work out the points of connection between what Mr. Glashan's formulæ would have been, if set right, and the general system now presented, but I have succeeded in doing so, and shall now present Mr. Glashan's system correctly, though in my own language.

32. If our general parameters, q, r, t, w, are made rational functions of four other parameters, g, k, m, n, which at the same time are rational functions of q, r, t, w, the new parameters must take the place of the old without lessening the generality of the system, and the group covered by them is the whole mass of resolvable quintics. If, on the other hand, we make q, r, t, w, rational functions of g, k, m, n, while the latter are not all rational functions of the former, we may assign rational values at will to g, k, m, n, and thereby always produce resolvable quintics, but our field is now restricted to a smaller group, outside of which are to be found all resolvable quintics produced by assigning rational values to q, r, t, w, which do not correspond to rational values of g, k, m, n. Mr. Glashan's parameters were g, k, m, n, of which q, r, t, w, are rational functions, but to obtain which from q, r, t, w, requires (except when $r = -t$) the intervention of a biquadratic equation. The relations between

* His words were: "the coefficients must be so related that *if* $p_2 = n\theta k^2$, $p_3 = \alpha\theta^2 k^3$, and $p_4 = \beta\theta^2 k^4$, *then* must $p_5 = 2(1+n)\gamma\theta^2 k^5$," etc., the coefficients being p_2 for our γ, p_3 for our δ, etc. I have italicized two words.

all lie in the even intervals to the right and left of the origin, in each of which there is an odd number of roots. Neumann,* slightly modifying Bessel's proof, shows that $J_n(x)$, $-\frac{1}{2} < n < \frac{1}{2}$, has an odd number of roots in the intervals in question. We have seen that if $\frac{2i-1}{2} < n < \frac{2i+1}{2}$, where it will be supposed that i is a positive integer or 0, the $\varkappa^{\text{th}}$ positive root of $J_n(x)$ lies between the $\varkappa^{\text{th}}$ positive roots of $J_{\frac{2i-1}{2}}(x)$ and $J_{\frac{2i+1}{2}}(x)$, and that $J_n(x)$ has but one root in this interval. When $i=0$, we have: *The $\varkappa^{th}$ root of $J_n(x)$, $-\frac{1}{2} < n < \frac{1}{2}$, lies between the $\varkappa^{th}$ root of $J_{-\frac{1}{2}}(x)$ and of $J_{\frac{1}{2}}(x)$. Since* $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$ *and* $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$, *the $\varkappa^{th}$ root of $J_n(x)$ lies between $\frac{2\varkappa-1}{2}\pi$ and $\varkappa\pi$*, i. e. *in the $\varkappa^{th}$ even quadrant.*

When $i=1$, we have: The $\varkappa^{\text{th}}$ root of $J_n(x)$, $\frac{1}{2} < n < \frac{3}{2}$, lies between $\varkappa\pi$ and the $\varkappa^{\text{th}}$ root of $J_{\frac{3}{2}}(x)$,

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right),$$

the first positive root of $J_{\frac{3}{2}}(x)$ is greater than π, the first positive root of $J_{\frac{1}{2}}(x)$, and it is readily seen $\left(\text{by substituting in } J_{\frac{3}{2}}(x) \text{ the values } x = \frac{\pi}{2}, \frac{3\pi}{2}; \pi, 2\pi, \text{etc.}\right)$ that the $\varkappa^{\text{th}}$ root of $J_{\frac{3}{2}}(x)$ lies in the $\varkappa+1^{\text{st}}$ odd quadrant. *Thus the $\varkappa^{th}$ positive root of $J_n(x)$, $\frac{1}{2} < n < \frac{3}{2}$, lies in the $\varkappa+1^{st}$ odd quadrant.*

In the same way we could go on determining comparatively narrow limits for the roots of $J_n(x)$, $n > \frac{3}{2}$, by determining the intervals of $\frac{\pi}{2}$ in which the roots of $J_{i+\frac{1}{2}}(x)$ lie. Unfortunately there seems to be no simple law governing the distribution† of the smaller roots of $J_{i+\frac{1}{2}}(x)$.

Very good approximations to the roots of $J_0(x)$, which we have seen lie one by one‡ in the even quadrants to the right and left of the origin, are given by the mid-points of the intervals in question.

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* "Theorie der Bessel'sche Functionen," pp. 65–6.

† M. P. Rudski (tome XVIII of the Mém. de la Soc. Royale de Liège), after a somewhat cumbrous analysis arrives at the result, which is readily seen from the tables to be erroneous, that the $\varkappa^{th}$ root of $J_{i+\frac{1}{2}}(x)$ (i integral) lies between $(i+2\varkappa-1)\frac{\pi}{2}$ and $(i+2\varkappa)\frac{\pi}{2}$.

‡ A fact noticed in the review of Gray and Mathews' "Bessel Functions," Math. Bull., May, 1896, p. 258.

The curve $y = J_n(x)$, to the right of the origin, will consist of an infinite number of arches which, as x increases, become flatter and flatter while near the origin, the curve has an infinite branch going off to infinity in the positive direction. When n passes through the value -1, the Γ-function in the expansion for $J_n(x)p$ changes sign and the branch at the origin goes off to infinity in the opposite direction. Thus it is clear that when n decreased from the value $-\frac{1}{2}$ to the value $-1-\varepsilon$, where ε is a small positive quantity, an odd number of roots disappeared at the origin. The positive roots of $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$ are: $x_1 = \frac{\pi}{2}$, $x_2 = \frac{3\pi}{2} \ldots - x_\kappa = \frac{2\kappa - 1}{2}\pi \ldots$ The interval $0, \frac{\pi}{2}$ is by III′ the first to become vacant, the root x_1 decreasing from the value $\frac{\pi}{2}$ to 0 while the root x_2 remains in the interval $\frac{\pi}{2}, \frac{3\pi}{2}$, so that but one positive root was lost when n decreased through -1. In the same way it is shown that *when n decreases through any negative integer, one positive root is lost at the origin and only one.* The negative roots of $J_n(x)$ being equal in absolute value to the positive roots, behave precisely like them, and consequently when n decreases through a negative integer, one and only one negative root is lost at the origin. Denoting by x_n the n^{th} positive root of $J_n(x)$, $n < 0$, a question that naturally presents itself is: how large can we let $-\kappa$ $(\kappa < 0)$ become and $J_{n+\kappa}(x)$ still have a single root in each of the intervals $0x_1$, x_1x_2, x_2x_3, etc. The preceding considerations enable us to answer this question at once. *If n and κ are negative,* the positive roots of $J_n(x)$ and $J_{n+\kappa}(x)$ will separate each other if* $-\kappa < E(-n+1)+n$.

§2.—*Determination of Intervals for Roots of $J_n(x)$.*

The theorems relative to the motion of the real roots of $J_n(x)$ as n decreases can be applied to some interesting questions concerning the distribution of these roots.

Bessel† showed that if we divide the axis of reals up into intervals of $\frac{\pi}{2}$, the first interval to the right of the origin being $0, \frac{\pi}{2}$, then the roots of $J_0(x)$

*If n and κ are positive, the corresponding theorem (Math. Bull., Mar. 1897, p. 212) is: If $\kappa \leq 2$ *the positive roots of $J_n(x)$ and $J_{n+\kappa}(x)$ separate each other.*

† Berlin. Abhandlungen (1824), §14, "Untersuchung des Theils der Planetarischen Störungen," etc.

Sturm deduced, by the methods developed in the memoir already referred to, the well-known theorem that $J_n(x)$ has, when n is real, an infinite number of real roots.

The series for $J_n(x)$ shows that the negative roots are numerically equal to the positive roots, so that in future we shall only speak of the latter.

§1.—*On the Variation of the Roots of* $J_n(x)$.

In the 10th vol. of the Math. Ann. (footnote, p. 137), Schläfli has shown that the positive roots of $J_n(x)$, n positive, increase with n.* When n is negative, the corresponding theorem is: *the positive roots of* $J_n(x)$, $n < 0$, *decrease with* n. Both statements can, of course, be embodied in the single theorem: *the positive roots of* $J_n(x)$ *decrease as* n *decreases.* To prove this, when n is negative we proceed as follows: The large roots of $J_n(x)$ are given approximately by the expression $x_0 = \frac{\pi}{4}(2n + 1 + 4p)$,† p taking on in succession all large integral values; so that if $n < 0$ decrease, x_0 will decrease. It is only necessary to note that $y = \sqrt{x} J_n(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} = \left(\frac{4n^2 - 1}{x^2} - 1\right) y = \phi(x, n)\, y,$$

and that $\phi(x, n)$, $n < 0$, as n decreases, increases for all values of x. Thus when $x_0 > 0$ decreases, all the smaller positive roots must decrease, for otherwise we should have two roots $J_{n-\kappa}(x)$ $(\kappa > 0)$ between two consecutive roots of $J_n(x)$, which by III is impossible.

Since, as n increases, the roots of $J_n(x)$ and $J_{-n}(x)$ are moving in opposite directions, they must pass each other, and this takes place when and only when n passes through an integral value; for it is only then that J_n and J_{-n} cease to be linearly independent of each other. Theorem III′ tells us that, *if* x_1 *denote the smallest positive root of* $J_n(x)$, $n < 0$, $J_{n-\kappa}(x)$ $(\kappa > 0)$ *cannot have more than one root between* 0 *and* x_1.

Let us now suppose that n, starting with the value $-\frac{1}{2}$, in which case

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

decrease, the positive roots of $J_n(x)$ will decrease.

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* For a proof of this theorem see p. 203. Another deduction of this theorem, which I shall extend to the case where n is negative, is given on page 212, Math. Bull., Mar. 1897.

† Gray and Mathews, "Bessel Functions," p. 40.

[1] or [2], although α or β might be singular points of [1] or [2]. A slight generalization of III, which will be useful, is the following:

III'. Let y_1 be any solution of [1] and x_1 be the root of y_1 which in the interval $\alpha\beta$ is nearest α, then *no solution of* [2] *can vanish more than once between* x_1 *and* α. This follows directly from I and II, for let us suppose that y_2, a solution of [2], did vanish more than once between x_1 and α, then by I, a solution $\bar{y}_2$ of [2], which vanishes at x_1, will have at least one root between x_1 and α, and by II so must y_1, which contradicts our hypothesis that x_1 is the root of y_1 nearest α in the interval $\alpha\beta$.

ON THE ROOTS OF BESSEL'S FUNCTIONS.

The first questions that will be considered relate to the real roots of the transcendental function $J_n(x)$, where it will be supposed that the index n is real. The case where the index is positive has already been treated in considerable detail in the Math. Bulletin, Mar. 1897, in the paper by Professor Bôcher, "On Certain Methods of Sturm and their Application to the Roots of Bessel's Functions."* We shall accordingly consider mainly the case where n is negative.

$J_n(x)$ is a particular solution of the homogeneous linear differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\,\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0,$$

and is defined by the series†

$$J_n(x) = \frac{x^n}{2^n\Gamma(1+n)}\left[1 - \frac{1}{1!\,(1+n)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,(1+n)(2+n)}\left(\frac{x}{2}\right)^4 - \right]$$

except when n is integral we have the linearly independent solution

$$J_{-n}(x) = \frac{x^{-n}}{2^{-n}\Gamma(1-n)}\left[1 - \frac{1}{1!\,(1-n)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,(1-n)(2-n)}\left(\frac{x}{2}\right)^4 - \right],$$

the general solution being, when n is not integral, of the form

$$y_n = aJ_n(x) + bJ_{-n}(x).$$

When n is an integer we have the important relation

$$J_{-n}(x) = (-1)^n J_n(x).$$

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* This paper will be referred to in future by Math. Bull., Mar. 1897.

† For other definitions, e. g. by complex integrals, see Gray and Mathews' "Bessel Functions," p. 59.

A somewhat generalized form of this theorem is the following:

I′. *If, within an interval ab (excluding the ends) the coefficients of the differential equation*

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

*are continuous, and if there exists a solution y_1 which does not vanish between a and b and such that its ratio to a linearly independent solution y_2 approaches zero as we approach each end of the interval, then y_2 vanishes once and only once between a and b.**

In the case of a regular singular point at a, y_1 would be the solution corresponding to the larger exponent of a.

II. *If, ϕ_1 and ϕ_2 being continuous real functions of the real variable x, $\phi_1 < \phi_2$, and if y_1 and y_2 both vanish at a point x_0 and satisfy respectively the equations*

$$\frac{d^2y}{dx^2} = \phi_1(x)\,y, \qquad [1]$$

$$\frac{d^2y}{dx^2} = \phi_2(x)\,y, \qquad [2]$$

then if y_2 has n roots to the right (left) of x_0, y_1 will have at least n roots there and the x^{th} root of y_2 will be greater (less) than the x^{th} root of y_1 from x_0.†

III. *In a certain interval $\alpha\beta$ of the x-axis, if $\phi_1(x)$ and $\phi_2(x)$ are continuous, and $\phi_1(x) < \phi_2(x)$, then between two successive*‡ *roots, lying in this interval of a solution of the equation*

$$\frac{d^2y}{dx^2} = \phi_1(x)\,y, \qquad [1]$$

there cannot lie more than one root of a solution of the equation§

$$\frac{d^2y}{dx^2} = \phi_2(x)\,y. \qquad [2]$$

Here it is to be noted that the continuity of ϕ_1 and ϕ_2 restricts the application of the theorem to an interval of the x-axis containing no singular point of either

* Math. Bull., Mar. 1897, p. 211 and footnote.

† Liouville, tom. I, p. 125.

‡ The ends of an interval $x_{\nu-1}x_\nu$ determined by two consecutive roots are not excluded. Cf. II, above.

§ Liouville, tom. I, p. 186.

On the Roots of the Hypergeometric and Bessel's Functions.

BY M. B. PORTER.

INTRODUCTION.

The questions discussed in this paper relate primarily to the real roots of Bessel's functions of negative order, the real roots of the convergents of the continued fraction for J_n/J_{n+1}, and the real roots of the hypergeometric series. The theorems obtained are applied to the problems of enumerating the imaginary roots of $J_n(x)$ and the roots of $F(\alpha, \beta, \gamma, x)$ between 0 and 1.

The methods employed throughout are those developed by Sturm in his first Memoir, on the real roots of functions defined by homogeneous linear differential equations of the second order, published in the first volume of Liouville's Journal. The theorems of the above-mentioned memoir, as enunciated by Sturm, are applicable only to intervals of the x-axis, the end points being included, in which there are no singular points of the differential equation. It is found, however, that in certain cases these theorems admit of a generalization to the case in which one or both of the ends of the interval in question are singular points of the differential equation. These generalized forms of Sturm's theorem are especially useful in dealing with the roots of the principal branches of the hypergeometric and Bessel's functions.

The following theorems of Sturm's memoir, which for convenience of reference will be stated at this point, are of constant application:

I. *p and q being continuous real functions of the real variable x, between two successive roots of a solution of the differential equation*

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

*will lie one and only one root of any linearly independent solution.**

*Liouville, tom. I, p. 125.

42. I add a suggestion for the benefit of those who may have occasion to employ (34) or (37) to determine whether or not a given numerical quintic is resolvable. The regular course would be to determine the numerical values of the coefficients of the sextic, and then to examine the sextic to see whether it has a rational root. Referring, for example, to the quintic (83), the regular course would be to frame the resolvent (85) and to examine it for a rational root. It is however usually easier to find the values (see paragraph 4) of $\phi(\tau) = \mathrm{AL} - 25\mathrm{C}^2$, for $\tau = -1$, $\tau = 0$, $\tau = 1$, etc., by reference to the known values of A, L, and C, as for example in (84), without actually determining the numerical values of the coefficients of the sextic. In (84), what we have to do, after dividing L by 8 and C by -4, is to find a value of τ which shall reduce to zero $\phi(\tau) = (3\tau^5 + 10\tau^4 + 20\tau^3 + 40\tau^2 - 10\tau + 4)(839\tau - 1398) - 50(7\tau^3 - 10\tau^2 + 9\tau + 22)^2$. The first of these two terms must in that event be positive, and must be divisible by 50, so that τ must be even, and we might thus be led speedily to $\tau = 2$. Proceeding regularly, however, we find $\phi(-1) = -3.30839$; $\phi(0) = -32.19.49$; $\phi(1) = -27.17.167$. Among the divisors there is no ascending sequence with the common difference 67, which is the coefficient of τ^6, but we perceive $-3, -2, -1$, also $1, 2, 3$. We next find $\phi(2) = 0$.

Postscript, *February* 7, 1898.—Critical Case F: $\gamma = -(\delta^2 + t^2 v) \div 2\delta t$. This case is normal as regards solution, but if we attempt to produce a resolvable quintic of this form we find l indeterminate, so that special measures become necessary. In fact, if we substitute $-2\gamma\delta t$ for $\delta^2 + t^2 v$ in (42) we have at once $l = -\gamma$, whence $q = -1$, which causes the denominator of the constructive expression for l in (39) to vanish. It is therefore necessary that the numerator, $t^2 w^2 - r^2 - r^2 w^2$, shall vanish also. To effect this we must assign a rational value to the usually irrational quantity m in the equation $w = 2m \div (m^2 - 1)$, and we must also limit r to the value $r = 2mt \div (m^2 + 1)$. We have therefore three available rational parameters, besides $q = -1$, namely, t, m, γ. Then $l = -\gamma$, $\delta = -(r + t)\gamma$, $v = \gamma^2 \div (1 + w^2)$, and the values of ε and ζ follow from (40) and (41). In this case v is a perfect square and $v^{\frac{1}{2}}, r_2$, and s_2 are rational. For example, let $t = q = -1$, $m = -2$, and $\gamma = 5$; then $w = -\frac{4}{3}$, $r = \frac{4}{5}$, $\delta = 1$, $v = 9$, $\varepsilon = 0$, $\zeta = -22$, and the quintic is $y^5 + 50y^3 + 10y^2 - 22 = 0$. Since this paper was read I have devised many resolvable numerical examples involving the suppression of ε, but my impression, conveyed in the text, that a general method for suppressing ε is probably impracticable, is not weakened.

The resolvent (37) being $\text{AL} - 25\text{C}^3 = 0$, we see that the work can be somewhat reduced by using $\frac{1}{4}\text{C}$ and $\frac{1}{5}\text{C}$ and $\frac{1}{5}\text{L}$, and we have this resolvent for the produced quintic (83):

$$67\tau^6 + 11196\tau^5 - 8500\tau^4 - 800\tau^3 - 46360\tau^2 - 2464\tau - 29792 = 0. \qquad (85)$$

According to (80), this also has $\tau = 2$, and dividing by $\tau - 2$, as before, and writing z_2 for τ, we have this second produced quintic:

$$67z_2^5 + 11330z_2^4 + 14160z_2^3 + 27520z_2^2 + 8680z_2 + 14896 = 0. \qquad (86)$$

The process may be continued indefinitely, but it becomes more difficult. The first produced quintic (83) might have been framed as an original quintic by assuming the following values for the parameters, besides $ba^{-1} = \frac{8}{5}$: $q = \frac{58}{51}$, $r = \frac{16}{153}$, $t = \frac{8}{5}$, $w = \frac{9}{65}$. The value of v is $\frac{3}{45}$, as may be verified by using the covariants (84) in either of the formulæ (38).

40. The sextic (85) is a function of the coefficients of the quintic (83), supplying seven equations from which it would appear that the five fundamental coefficients of (83) may be obtained in terms of, and as rational functions of, the coefficients of the sextic. We might therefore work backwards, starting from the quintic (86), and multiplying it by $z_2 - 2$ to produce the sextic (85), then using the coefficients of the latter to determine those of the quintic (83). Similarly, from (83), knowing its resolvent 2, we could produce the sextic (82), from which the coefficients of the quintic (81) could be obtained by the solution of seven equations. In like manner it would seem that we could work backwards from our original quintic (81), multiplying it by $x - 2$ to produce a sextic, from which the coefficients of another and, so to speak, prior quintic could be ascertained, the chain of resolvable quintics thus produced one from the other extending in both directions without apparent limit.

41. The property just discussed, possessed by the sextic (37) in τ, cannot reasonably be looked for in connection with any other resolvent. It depends upon the relation (80) by which the rational resolvent τ_6 is the same function of the other roots of the sextic as it is of the roots of the original quintic, and this can only happen when, as is the case with τ, the function is of weight 1. It is not to be presumed that any other resolvent function of weight 1 exists which is possessed of this peculiar property.

that (78) can be collected at once into this form (also an alternating function of the roots):

$$\mathrm{C}_6\tau_1(\tau_3-\tau_2)\,.\,\tau_6-\mathrm{C}_6\tau_1(\tau_3-\tau_2)\,\tau_4=0\,, \tag{79}$$

where the sequence 13254 applies to the subscripts of τ. Therefore,

$$\tau_6=\frac{\mathrm{C}_6x_1(x_3-x_2)\,x_4}{\mathrm{C}_6x_1(x_3-x_2)}=\frac{\mathrm{C}_6\tau_1(\tau_3-\tau_2)\,\tau_4}{\mathrm{C}_6\tau_1(\tau_3-\tau_2)}\,. \tag{80}$$

This means that τ_6 is the same function of the other roots of my resolvent (37) as it is of the roots of the quintic. In other words, this remarkable theorem is true, that if the sextic resolvent in τ have a rational root τ_6, and if the sextic be reduced to a quintic by dividing it by $\tau-\tau_6$, the resolvent of the new quintic will itself have the same root τ_6.*

39. There are therefore framed implicitly an apparently infinite number of resolvable quintics for each one which we may frame directly by means of the method of (39) and (40), the new series being produced one from another by the formation and, so to speak, decapitation of resolvent sextics of the form (37). As an example, we may put $q=0$, $r=-2$, $t=2$, $w=3$, so that from (39) and (40), or from (69), we have $l=1$, $v=\frac{1}{10}$, $\varepsilon=\frac{3}{2}$, $\zeta=-2$, so that the quintic constructed is

$$x^5+\tfrac{15}{2}x-2=0\,. \tag{81}$$

Employing the general resolvent (37), or preferably the reduced form (68) which it assumes in this trinomial case, the resolvent for (81) is, after clearing of fractions,

$$3\tau^6+4\tau^5-90\tau^2+24\tau-8=0\,. \tag{82}$$

Dividing this by $\tau-2$, and writing z for τ, we have the produced quintic

$$3z^5+10z^4+20z^3+40z^2-10z+4=0\,. \tag{83}$$

The necessary covariants of this, for developing the sextic resolvent according to (37), are

$$\left.\begin{aligned}&\mathrm{A}=3\tau^5+10\tau^4+20\tau^3+40\tau^2-10\tau+4,\\&\mathrm{L}=6712\tau-11184,\\&\mathrm{C}=-28\tau^3+40\tau^2-36\tau-88\,.\end{aligned}\right\} \tag{84}$$

* As a fact ascertained from a sufficient number of numerical examples, but as yet devoid of proof, this theorem was laid before this Society by the writer on taking his seat as President in December, 1890; and was also, shortly afterwards, communicated in a private letter to Professor Cayley.

is true of all transpositions not affecting x_4. Since the other grand terms are derived from the first by successive cyclic substitutions according to the sequence $x_1x_3x_2x_5x_4$, any transposition affects every grand term only by a change of sign, except transpositions including x_4 for the first grand term, x_1 for the second, x_3 for the third, x_2 for the fourth, and x_5 for the fifth. Any single transposition, say (x_1x_4), affects three grand terms only by a change of sign, and we shall now see that, as regards the other two grand terms, it merely transforms each into the other, again with a change of sign. For (x_1x_4) changes ψ_1 into ψ_2, ψ_2 into ψ_1, ψ_3 into ψ_5, ψ_4 into ψ_6, ψ_5 into ψ_3, and ψ_6 into ψ_4, in each case with a change of sign, and the same effects are produced upon the functions ϕ. Hence (x_1x_4) transforms the first grand term $(\psi_1\phi_3 - \psi_3\phi_1)(\psi_2\phi_4 - \psi_4\phi_2)(\psi_5\phi_6 - \psi_6\phi_5)$ into the second grand term with its sign changed, namely, $-(\psi_2\phi_6 - \psi_6\phi_2)(\psi_4\phi_3 - \psi_3\phi_4)(\psi_1\phi_6 - \psi_6\phi_1)$, and *vice versa*, so that (x_1x_4) alters (76) only by changing its sign. The cyclic substitution $(x_1x_3x_2x_5x_4)$ leaves (76) unaltered, so that successively we have, after (x_4x_1) the transpositions (x_1x_3), (x_3x_2), (x_2x_5), (x_5x_4), all affecting (76) only by a change of sign; and since all possible transpositions are composed of odd numbers of the transpositions named, any single transposition can affect (76) only by a change of sign. It should be remarked that the sequence 13254 in the roots is equivalent to 12435 in the subscripts of ϕ and ψ.

38. By expanding and rearranging the terms of (76) we have, after dividing throughout by 2,

$$\begin{aligned}
&\psi_1\psi_3\psi_6\phi_2\phi_4\phi_5 - \psi_1\psi_2\psi_6\phi_3\phi_4\phi_5 \\
+\, &\psi_3\psi_2\psi_6\phi_1\phi_4\phi_5 - \psi_3\psi_5\psi_6\phi_1\phi_2\phi_4 \\
+\, &\psi_2\psi_5\psi_6\phi_1\phi_3\phi_4 - \psi_2\psi_4\psi_6\phi_1\phi_3\phi_5 \\
+\, &\psi_5\psi_4\psi_6\phi_1\phi_2\phi_3 - \psi_5\psi_1\psi_6\phi_2\phi_3\phi_4 \\
+\, &\psi_4\psi_1\psi_6\phi_2\phi_3\phi_5 - \psi_4\psi_3\psi_6\phi_1\phi_2\phi_5 \\
+\, &\psi_1\psi_2\psi_4\phi_3\phi_5\phi_6 - \psi_1\psi_3\psi_4\phi_2\phi_5\phi_6 \\
+\, &\psi_3\psi_5\psi_1\phi_2\phi_4\phi_6 - \psi_3\psi_2\psi_1\phi_4\phi_5\phi_6 \\
+\, &\psi_2\psi_4\psi_3\phi_1\phi_5\phi_6 - \psi_2\psi_5\psi_3\phi_1\phi_4\phi_6 \\
+\, &\psi_5\psi_1\psi_2\phi_3\phi_4\phi_6 - \psi_5\psi_4\psi_2\phi_1\phi_3\phi_6 \\
+\, &\psi_4\psi_3\psi_5\phi_1\phi_2\phi_6 - \psi_4\psi_1\psi_5\phi_2\phi_3\phi_6 = 0.
\end{aligned} \tag{78}$$

Substituting $\tau\phi$ for ψ, and dividing throughout by $\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6$, we perceive

36. Let us take $x_1^6x_2^5x_3^3x_4$ as the term in (76) whose coefficient is to be examined. Now (76) consists of five grand terms, the first of which is $(\psi_1\phi_3-\psi_3\phi_1)$ $(\psi_2\phi_4-\psi_4\phi_2)(\psi_5\phi_6-\psi_6\phi_5)$, and the others are derived from this by successive repetition of the cyclic substitution $x_1x_3x_2x_5x_4$. The coefficient of $x_1^6x_2^5x_3^3x_4$ in the second grand term is therefore the same as that of $x_4^6x_3^5x_1^3x_5$ in the first; in the third is the same as that of $x_5^6x_1^5x_4^3x_2$ in the first; in the fourth the same as that of $x_2^6x_4^5x_5^3x_3$, and in the fifth as that of $x_3^6x_5^5x_2^3x_1$. The sum of the coefficients of these expressions in the first grand term is therefore the coefficient desired from the whole of (76) for $x_1^6x_2^5x_3^3x_4$. The three factors of which the first grand term is composed consist each of 24 terms, of which I quote the following (all to be multiplied by 2) as bearing on the present example: in the first factor, $-x_1^2x_3x_4^2$, $x_1x_3^2x_4^2$, $-x_2^2x_3^2x_5$, $-x_2^2x_4^2x_5$, $2x_1^2x_2x_3x_4$; in the second factor, $-x_1^2x_2^2x_3$, $-x_1x_3^2x_5^2$, $x_2x_3^2x_5^2$, $-x_3^2x_4^2x_5$, $2x_2^2x_3x_4x_5$; and in the third factor, $x_1^2x_2^2x_3$, $x_1^2x_3x_4^2$, $-x_1x_3^2x_4^2$, $-x_1x_3^2x_5^2$, $-x_2^2x_4^2x_5$, $x_2x_3^2x_5^2$. We find that $2x_1^2x_2x_3x_4(-x_1^2x_2^2x_3)\,x_1^2x_2^2x_3=-2x_1^6x_2^5x_3^3x_4$; that $-x_1^2x_3x_4^2(-x_3^2x_4^2x_5)(-x_1x_3^2x_4^2)=-x_4^6x_3^5x_1^3x_5$, and $x_1x_3^2x_4^2(-x_3^2x_4^2x_5)(x_1^2x_3x_4^2)$ $=-x_4^6x_3^5x_1^3x_5$; that there is no term $x_5^6x_1^5x_4^3x_2$; that $-x_2^2x_4^2x_5(2x_2^2x_3x_4x_5)(-x_2^2x_4^2x_5)$ $=2x_2^6x_4^5x_5^3x_3$; and that $-x_2^2x_3^2x_5(-x_1x_3^2x_5^2)\,x_2x_3^2x_5^2=x_3^6x_5^5x_2^3x_1$, and $-x_2^2x_3^2x_5\,.\,x_2x_3^2x_5^2$ $(-x_1x_3^2x_5^2)=x_3^6x_5^5x_2^3x_1$. The sum of the coefficients is zero, and there is therefore in (76) no term $x_1^6x_2^5x_3^3x_4$.

37. It remains to be shown that (76) merely changes sign when any two roots are transposed. Let us consider first the first grand term $(\psi_1\phi_3-\psi_3\phi_1)$ $(\psi_2\phi_4-\psi_4\phi_2)(\psi_5\phi_6-\psi_6\phi_5)$, and let us begin by excluding the root x_4 from transposition. In the following schedule we have the effects produced upon the several functions ψ by the transpositions noted at the left, and it is to be observed that the same effects, as to both signs and subscripts, are produced upon the corresponding functions ϕ.

$$\left.\begin{array}{lcccccc} & \psi_1 & \psi_3 & \psi_2 & \psi_4 & \psi_5 & \psi_6 \\ \hline (x_1x_2) & -\psi_3 & -\psi_1 & -\psi_6 & -\psi_5 & -\psi_4 & -\psi_2 \\ (x_1x_3) & -\psi_5 & -\psi_6 & -\psi_4 & -\psi_2 & -\psi_1 & -\psi_3 \\ (x_1x_5) & -\psi_4 & -\psi_2 & -\psi_3 & -\psi_1 & -\psi_6 & -\psi_5 \end{array}\right\} \qquad (77)$$

Each of the transpositions (x_1x_2), (x_1x_3), (x_1x_5), therefore, affects the first grand term only by changing its sign, and as all other transpositions of x_1, x_2, x_3, x_5, may be transformed into an odd number of the transpositions named, the same

35. I say now that

$$c_6(\psi_1\phi_3-\psi_3\phi_1)(\psi_2\phi_4-\psi_4\phi_2)(\psi_5\phi_6-\psi_6\phi_5)=0. \tag{76}$$

It will shortly be shown that it is an alternating function, that is to say, one which has the same value, but with the opposite sign, whenever any two roots are transposed one for the other. As such, being of the fifteenth degree in the roots, it must be composed of two factors, one namely of the tenth degree, consisting of the product of the differences of the roots, $(x_1-x_2)(x_1-x_3)\ldots(x_4-x_5)$, which changes sign when a transposition occurs, the other a symmetrical function of the fifth degree, upon which a transposition produces no effect. If, having this in mind, and expecting the product to consist of a large number of terms, we attempt to evaluate (76) we are met nevertheless by a zero coefficient for the first term we deal with, and again for the next term, whatever it may be, and so on, until we cannot but be convinced that the symmetrical factor must vanish, if it exists at all even as a phantom. To evaluate the myriads of terms contained in the product would be impossible. It is, however, feasible to reduce greatly the number of terms which it is necessary to examine. Every term found non-existent proves the non-existence of every other term of the same form with different subscripts, since transposition leaves the value unchanged unless in sign. Each of the three factors of (76) consists of terms in which no one root enters in a power above the second, so that no term of the product can contain any power of one root above the sixth. The three factors of (76) contain 24 terms each, which when examined show that the product can contain no term of three letters, and that no term can, as to its literal part, be a perfect cube. Each factor is of the form $\psi_m\phi_n-\psi_n\phi_m$, and every term of $\psi_m\phi_n$ is represented in $\psi_n\phi_m$ by a complementary term affected by the same coefficient, the literal parts being such that the product of the two is $x_1^6x_2^6x_3^6x_4^6x_5^6$; and it follows that when any term in the product is found to vanish, the complementary term must likewise vanish. For example, $x_1^5x_2^4x_3^4x_4^2$ is complementary to $x_1x_2^2x_3^2x_4^4x_5^6$, equivalent as a form to $x_1^6x_2^4x_3^2x_4^2x_5$. For the same reason there can be no term $x_1^3x_2^3x_3^3x_4^3x_5^3$, which is besides to be set aside as a cube. It thus results that the number of terms in the product which, to make sure that every term vanishes it is necessary to examine, and which I have examined, is reduced to sixteen. It is unnecessary to present the details, but to assist any one who may wish to look into the matter I illustrate the process of evaluation by an example.

33. Thus far we have dealt almost altogether with the shorter form (2) of the general quintic, for which my resolvent is the quantity t. The broader resolvent in τ was exhibited in (37) as the covariant sextic $\text{AL} - 25\text{C}^2 = 0$. It will be remembered that ϕ is the same function of x, in the longer form (1) of the general quintic, as it is of y in the shorter form (2), but that in (1) we have $\psi = t\phi - ba^{-1}\phi = \sigma - ba^{-1}\phi$, and $\tau = \psi\phi^{-1} = t - ba^{-1}$, just as $x = y - ba^{-1}$. It has also been remarked that for the longer form of the general quintic the quantity ba^{-1} may be regarded in a sense as a fifth parameter, though it is much broader as a function of the roots than the four parameters, having only one value instead of six. Let us now examine the form of the six values of τ in detail, τ being the leading parameter of the general quintic (1), reduced to t when $ba^{-1} = 0$, as in the shorter quintic (2).

34. In paragraph 13 we had the symbol C—cycle or circle—so defined that $\text{C}f(x_1, x_2, x_3, x_4, x_5) = f(x_1, x_2, x_3, x_4, x_5) + f(x_2, x_3, x_4, x_5, x_1) + f(x_3, x_4, x_5, x_1, x_2) + f(x_4, x_5, x_1, x_2, x_3) + f(x_5, x_1, x_2, x_3, x_4)$; that is to say, C represented the sum of five similar functions of the roots, each function differing from the one preceding by a change in the subscripts, from 1 to 2, and so on, in the order 12345. Let us hereafter write C_1 for C as heretofore used in this sense. For the family of six-valued functions of the roots with which we are concerned, there are five other possible sequences, which may be distinguished by assigning different subscripts to C,* thus:

$$\left.\begin{array}{ll} \text{C}_1 : 12345 & \text{C}_4 : 12453 \\ \text{C}_2 : 12534 & \text{C}_5 : 13425 \\ \text{C}_3 : 14235 & \text{C}_6 : 13254 \end{array}\right\} \qquad (75)$$

We have had $\phi_1 = \text{C}_1 x_1 (x_2 - x_3)$, and $\psi_1 = \text{C}_1 x_1 (x_2 - x_3) x_5$. Similarly, $\phi_2 = \text{C}_2 x_1 (x_2 - x_5)$, $\psi_2 = \text{C}_2 x_1 (x_2 - x_5) x_4$; $\phi_3 = \text{C}_3 x_1 (x_4 - x_2)$, $\psi_3 = \text{C}_3 x_1 (x_4 - x_2) x_5$; and so on. It will be observed that each ϕ consists of five positive products, each of two adjacent roots, according to the sequence employed, plus five negative products, each of two non-adjacent roots. Similarly, each ψ consists of five positive products, each of three adjacent roots, plus five negative products, each of two adjacent and one non-adjacent roots. Then the six values of τ are $\tau_1 = \psi_1\phi_1^{-1}$; $\tau_2 = \psi_2\phi_2^{-2}$; and so on.

* The subscripts were arranged differently in the paper of 1885, following Cayley. It is necessary for symmetry to make certain changes.

the general parameters q, r, t, w, underlying all resolvable quintics, and Mr. Glashan's parameters g, k, m, n, pertaining to a group only, are:

$$\left.\begin{aligned} t &= \theta k(\alpha\lambda - 2), \\ q &= n\mu, \\ r &= \theta k(\alpha\mu - \alpha\lambda + 2), \\ w &= 2\,\frac{\alpha\eta(m-g) - m}{\alpha\eta(\alpha\lambda - 2) + 2}, \end{aligned}\right\} \tag{74}$$

where $\eta = (1+n)(1+m^2)$, $\theta = 1 + n - n^2\eta$, $\lambda = 1 + m^2 - g^2 - ng^2$, $\alpha = 4g(m - g - ng) \div (\lambda^2 - 1 - m^2)$, and $\mu = [2\alpha\eta(\alpha\lambda - 2) + 4] \div [\alpha^2\eta + (\alpha\lambda - 2)^2(1+n) - 4n]$. It will be seen that t, q, r, w, are all rational when g, k, m, n, are rational. Substituting these values in (39) we have $l = \mu^{-1}\theta k^2$, and it may also be proved by due substitution that $\mu^2(1 + w^2) = 1 + m^2$. Hence, from (40), $\gamma = n\theta k^2$, $\delta = \alpha\theta^2k^3$, $v = \theta^2k^4 \div (1 + m^2)$. Then ε and ζ may be determined by (40) and (41). For example, let the group parameters be $g = -1$, $k = \frac{1}{4}$, $m = 2$, $n = 1$; then $\eta = 10$, $\theta = -8$, $\lambda = 3$, $\alpha = -4$, $\mu = \frac{281}{187}$, and by (74) the general parameters are $t = 28$, $q = \frac{281}{187}$, $r = -\frac{1568}{187}$, $w = -\frac{122}{281}$. We find that $\gamma = -\frac{1}{2}$, $\delta = -4$, $v = \frac{1}{20}$, $\varepsilon = 34$. It has been said that to ascertain the values of g, k, m, n, from those of t, v, and the coefficients, it is necessary in general to find one of the roots of a biquadratic equation: in the present example the biquadratic must have a rational root. The quantities t, v, q, r, w, l, p, r_1, r_2^2, s_1, s_2^2, all belong to one family of functions, conjugate to which are five other like families, since each of these quantities is a six-valued function of the roots of the quintic. When we find, by means of the resolvent, a rational value of t, we are enabled, by means of the formulæ herein given, to obtain the corresponding values of the other functions comprised in the same family, and we have, for present purposes, nothing to do with the other five sets of values. The various quantities contained in one family with the given value of t are each and all rational functions of t and of each other, and they have therefore been spoken of herein as "rational." It is unnecessary to remark that the relations between them hold good, and that they are all still rational functions, each of each other, when the value of t is not rational, and when therefore all other quantities of the same family are in general irrational. The quantities g, k, m, n, discussed by Mr. Glashan, belong to one subdivision of this family of functions.

§3.—*On the Roots of* $J_n(x)$ *and* $J_{n+i}(x)$.

When i is integral, and in this section we shall suppose it so, $J_n(x)$ and $J_{n+i}(x)$ are kindred (*verwandt*) functions and relations of the form*

$$J_{n+i}(x) = g^{(n)}_{i-1}(x)\, J_{n+1}(x) - g^{(n)}_{i-2}(x)\, J_n(x)$$

exist where the g's are rational integral functions of $\frac{2}{x}$. Such relations are only a limiting case of a general class of similar relations connecting three kindred hypergeometric functions.

The roots† of these rational functions play an important rôle in the study of the roots of $J_n(x)$. The first case to be considered is where $i = \pm 1$, and here we have the theorem:‡ *The positive roots of* $J_{n\pm 1}(x)$ *and* $J_n(x)$ *separate each other.* This follows at once from the two formulæ

$$\frac{d}{dx}\{x^{-n}J_n(x)\} = -x^{-n}J_{n+1}(x), \tag{1}$$

$$\frac{d}{dx}\{x^{n+1}J_{n+1}(x)\} = x^{n+1}J_n(x). \tag{2}$$

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* Gray and Mathews' "Bessel Functions," p. 13.

† Hurwitz, "Ueber die Nullstellen der Bessel'sche Function," Math. Ann., Bd. 33 (1888).

‡ Van Vleck (vol. XIX, p. 75, American Journal) first established this theorem by means of the relation

$$J_{n+1}J_{-n} - J_{-n-1}J_n = \frac{2\sin(n+1)\pi}{\pi x}\left(n \text{ integral } J_{n+1}Y_n - J_nY_{n+1} = \frac{1}{x}\right),$$

extending the method to the contiguous P-functions of Riemann by means of the analogous relation connecting two pairs of linearly independent contiguous P-functions. In the Math. Bull., Mar. 1897, this theorem is proved by methods more analogous to those here used. Van Vleck's method can be applied to similar questions concerning the real roots of any solution y of the homogeneous linear differential equation

$$y'' + py' + qy = 0, \tag*{[1]}$$

and the real roots of y' and y''. Here we make use of Abel's relation (α) $y_1y_2' - y_2y_1' = -Ae^{-\int p dx}$, and its derivative (β) $y_1y_2'' - y_2y_1'' = -Ape^{-\int p dx}$, where y_1 and y_2 denote linearly independent solutions of [1]. The functions y_1' and y_1'' satisfy a homogeneous linear differential equation of the second order with the singular points of [1] and, in general, certain accessory singular points besides. Equations (α) and (β), applied to Bessel's equation, show that *if* $|x| > |n|$ *the real roots of either* $J_n'(x)$ *or* $J_n''(x)$ *and of* $J_n(x)$ *separate each other.* This theorem was given in the Math. Bull., Mar. 1897, p. 207. It is to be noted that the theorems thus obtained do not, in general, relate to corresponding branches of contiguous functions. Applied to any solution y of the hypergeometric differential equation

$$y'' - \frac{\gamma - (\alpha+\beta+1)x}{x(1-x)}y' - \frac{\alpha\beta}{x(1-x)}y = 0 \tag*{[2]}$$

(α) yields: *In any interval of the x-axis containing no singular point of* (2), *the roots of* $\frac{d^ry}{dx^r}$ *and* $\frac{d^{r+1}y}{dx^{r+1}}$ *separate each other.* And in the same way, *the roots of* $P^n_m(x)$ *and* $P^{n+1}_m(x)$ *between* $+1$ *and* -1 *separate each other* ($P^n_m(x)$ denoting the associated function of Spherical Harmonics).

(1) by Rolle's theorem shows that between two consecutive positive roots of $J_n(x)$ lies at least one root of $J_{n+1}(x)$, and (2) that between two consecutive positive roots of $J_{n+1}(x)$ lies at least one root of $J_n(x)$, hence the theorem. In the same way it can be shown that the positive roots of

$$y_n = a\,J_n(x) + bJ_{-n}(x) \tag{3}$$

and of

$$-aJ_{n+1}(x) + bJ_{-n-1}(x) \tag{4}$$

separate each other. It is to be noted that (1) and (2) are not contiguous unless a or b vanish or n is integral.

Attention has already been called to the relation

$$J_{n+i}(x) = g_{i-1}^{(n)}(x)\,J_{n+1}(x) - g_{i-2}^{(n)}(x)\,J_n(x).$$

The following lemma of constant application will show the use that can be made of the roots of $g_{i-1}^{(n)}(x)$ in determining the relative positions of the roots of $J_n(x)$ and $J_{n+i}(x)$. Let x_j and x_{j+1} be two consecutive positive roots of $J_n(x)$, we have

$$J_{n+i}(x_j) \quad = g_{i-1}^{(n)}(x_j)\,J_{n+1}(x_j)$$

and

$$J_{n+i}(x_{j+1}) = g_{i-1}^{(n)}(x_{j+1})\,J_{n+1}(x_{j+1}).$$

Since neither x_j nor x_{j+1} is zero, neither $J_{n+1}(x_j)$ nor $J_{n+1}(x_{j+1})$ can vanish. We know that $J_{n+1}(x)$ changes sign once between x_j and x_{j+1}, therefore unless $g_{i-1}^{(n)}(x)$ change sign an odd number of times between x_j and x_{j+1}, $J_{n+i}(x)$ will have an odd number of roots between x_j and x_{j+1}. If $|h+i| > (n)$, $J_{n+i}(x)$ can have at most one root between x_j and x_{j+1}, by theorem III of Sturm, p. 194. Thus from the relations,

$$J_{n+2}(x) = \frac{2(n+1)}{x}\,J_{n+1}(x) - J_n(x), \qquad n > 0$$

$$J_n(x) \quad = \frac{2(n-1)}{x}\,J_{n-1} - J_{n-2}(x), \qquad n < 0$$

we get the general theorem: *The positive roots of $J_{n\pm 2}(x)$ and $J_n(x)$ separate each other.**

§4.—*On the Functions $G_i^{(n)}(x)$.*

The rational functions that we have denoted above by $g_{i-1}^{(n)}(x)$ and $g_{i-2}^{(n)}(x)$ are computed at once from the relations *inter contiguas.* We have

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*Math. Bull., Mar. 1897, p. 207.

$$J_{n+i}(x) = \frac{2(n+i-1)}{x} J_{n+i-1}(x) - J_{n+i-2},$$
$$J_{n+i-1} = \frac{2(n+i-2)}{x} J_{n+i-2} - J_{n+i-3},$$
$$J_{n+i-2} = \frac{2(n+i-3)}{x} J_{n+i-3} - J_{n+i-4},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$J_{n+3} = \frac{2(n+2)}{x} J_{n+2} - J_{n+1},$$
$$J_{n+2} = \frac{2(n+1)}{x} J_{n+1} - J_{n}.$$

Thus
$$J_{n+i} = g_{\kappa}^{(n)} J_{n+i-\kappa} - g_{\kappa-1}^{(n)}(x) J_{n+i-\kappa-1} = g_{i-1}^{n}(x) J_{n+1} - g_{i-2}^{n}(x) J_{n},$$

where
$$g_{\kappa}^{(n)}(x) = \frac{2(n+i-\kappa)}{x} g_{i-1}^{(n)}(x) - g_{i-2}^{(n)}(x) \qquad [I]$$

and
$$g_{-1}^{(n)}(x) = 0, \quad g_{0}^{(n)}(x) = 1,$$

thus $g_{\kappa}^{(n)}(x)$ is a polynomial in $\frac{2}{x}$ of degree κ.

From the recurring relation I we get

$$g_{1}^{(n)}(x) = \frac{2(n+i-1)}{x},$$
$$g_{2}^{(n)}(x) = \frac{2^2}{x^2}(n+i-2)(n+i-1) - 1,$$
$$g_{3}^{(n)}(x) = \frac{2^3}{x^3}(n+i-3)(n+i-2)(n+i-1) - \frac{2}{x}\, 2(n+i-2),$$

and by inspection,

$$g_{\kappa}^{(n)}(x) = \frac{2^{\kappa}}{x^{\kappa}}(n+i-\kappa)(n+i-\kappa+1)\ldots\ldots(n+i-2)(n+i-1),$$
$$-\frac{2^{\kappa-2}}{x^{\kappa-2}}\frac{(\kappa-1)}{1!}(n+i-\kappa+2)\ldots\ldots(n+i-2),$$
$$+\frac{2^{\kappa-4}}{x^{\kappa-4}}\frac{(\kappa-2)(\kappa-3)}{2!}(n+i-\kappa+3)\ldots\ldots(n+i-3),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

the r^{th} term being

$$+(-1)^{r-1}\frac{2^{\kappa-2r+2}}{x^{\kappa-2r+2}}\,\frac{(\kappa+1-r)(\kappa-r)\dots(\kappa-2r+3)}{(r-1)!}\,(n+i-\kappa+r-1)\dots(n+i-r).$$

Since the expression $\dfrac{(\kappa+1-r)(\kappa-r)\dots\dots(\kappa-2r+3)}{(r-1)!}$ is always a positive integer, the sign of the r^{th} term depends only on the factor

$$(n+i-\kappa+r-1)\dots\dots(n+i-r),$$

moreover, the coefficients of the various powers of $\dfrac{2}{x}$ are rational integral functions of n and are therefore rational numbers if n is rational.

$$J_{n+i}(x)=g_{i-1}^{(n)}(x)\,J_{n+1}(x)-g_{i-2}^{(n)}(x)\,J_n(x).$$

If $J_n(x)$ and $J_{n+i}(x)$ both vanish for the same value of x as $x_0 \neq 0$, we must have $g_{i-1}^{(n)}(x_0)=0$,* i.e. if n is rational or merely algebraically irrational, the common root x_0 must be algebraically irrational. If we knew that the positive roots of $J_n(x)$ were, when n is rational or algebraically irrational, transcendental numbers, we could at once conclude that $J_n(x)$ and $J_{n+i}(x)$ have no positive root in common.†

When n is the half of an odd integer it is easy to show that the roots of $J_n(x)$ are transcendental numbers.

$$J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sin x,$$

$$J_{\frac{3}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left\{\frac{\sin x}{x}-\cos x\right\},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$J_{i+\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left\{g_i^{(\frac{1}{2}+\frac{1}{2})}(x)\sin x-g_{i-1}^{(\frac{1}{2}+\frac{1}{2})}(x)\cos x\right\}.$$

The coefficients of the various powers of $\dfrac{2}{x}$ in the g-functions are rational numbers, and it is clear from the relation (I), p. 201, that $g_\kappa^{(n)}(x)$ and $g_{\kappa-1}^{(n)}(x)$ can-

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* Since $J_n(x)$ and $J_{n+1}(x)$ have no root in common save 0.

† Bourget, Ann. de l'École Normale, 1866, p. 66, stated such a theorem when n is integral without proof.

not both vanish for the same value of x. Let $x_0 \neq 0$ be a root of $J_{i+\frac{1}{2}}(x)$, and suppose that $g_i^{(i+\frac{1}{2})}(x_0) \neq 0$, we have

$$\tan x_0 = \frac{g_{i-1}^{(i+\frac{1}{2})}(x_0)}{g_i^{(i+\frac{1}{2})}(x_0)} = a, \tag{III}$$

where, if x_0 is merely algebraically irrational so is a. III becomes in terms of exponentials

$$e^{2x_0 i} = \frac{1+ai}{1-ai}.$$

Lindemann's equation,* from which he deduced the transcendental irrationality of π, asserts that if x_0 is algebraically irrational, this equation is impossible. Thus: *No two Bessel's functions whose orders are the halves of odd integers can have a positive root in common.*

The recurring relation connecting three consecutive g-functions suggests† a useful expression of $g_\kappa^{(n)}(x)/g_{\kappa-1}^{(n)}(x)$ as a continued fraction.

$$g_{i-1}^{(n)}(x)/g_{i-2}^{(n)}(x) = \frac{2(n+1)}{x} - \cfrac{1}{\cfrac{2(n+2)}{x} - \cdots - \cfrac{1}{\cfrac{2(n+i-1)}{x}}}.$$

This is evidently the $i-1^{\text{st}}$ convergent of the continued fraction

$$\frac{J_n(x)}{J_{n+1}(x)} = \frac{2(n+1)}{x} - \cfrac{1}{\cfrac{2(n+2)}{x} - \cfrac{1}{\cfrac{2(n+3)}{x} - \text{etc.}}}$$

Writing
$$g_{i-1}^{(n)}(x)/g_{i-2}^{n}(x) = \frac{2(n+1)}{x} - \phi(x, n)$$

from the continued fraction itself, it is easily seen that if, when n and x are positive, we slightly increase n, x remaining constant, the term $\frac{2(n+1)}{x}$ increases while $\phi(x, n)$ decreases, thus: *when n increases, the positive roots of $g_{i-1}^{(n)}(x)$ increase, if $n > 0$.*

It is easily shown that the roots of $g_{i-1}^{(n)}(x)$, as i increases indefinitely, cluster‡ about the roots of $J_n(x)$, so that we get a new proof of the theorem§ that the positive roots of $J_n(x)$, $n > 0$, increase with n.

* Math. Ann., Bd. 20, p. 224.

† Hurwitz, Math. Ann., Bd. 33, l. c.

‡ Hurwitz, Math. Ann., Bd. 33, pp. 250–52.

§ Quoted on p. 196.

On page 200 it was seen that the roots of $g_{i-1}^{(n)}(x)$ can be used to find those intervals determined by consecutive positive roots of $J_n(x)$ in which no root of $J_{n+i}(x)$ lay. From the expression for $g_{i-1}^{(n)}(x)$ on page 201 we see that $g_{i-1}^{(n)}(x)$ has at most $E\left(\frac{i-1}{2}\right)$ positive roots. It is shown (p. 212, Math. Bull., Mar. 1897) that, *if n is positive and $2p < \kappa \leqq 2p+2$ where p is any positive integer, in each of the intervals* bounded by successive positive roots of $J_n(x)$, there will be one and only one root of $J_{n+\kappa}(x)$, except in p of these intervals, in which there will be no root of* $J_{n+\kappa}(x)$. Thus in the case we have just been considering there will be $E\left(\frac{i-1}{2}\right)$ vacant intervals, and consequently, *if n and i are positive, all the roots of $g_{i-1}^{(n)}(x)$ are real and, moreover, at most one lies in any interval determined by two consecutive positive roots of $J_n(x)$.*† When both n and i are negative we have a similar problem and one whose solution is deducible directly from the case in which n and i are positive. We have

$$\begin{aligned} J_{n+i} &= \frac{2(n+i+1)}{x} J_{n+i+1} - J_{n+i+2}, \\ J_{n+i+1} &= \frac{2(n+i+2)}{x} J_{n+i+2} - J_{n+i+3}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ J_{n-2} &= \frac{2(n-1)}{x} J_{n-1} - J_n, \end{aligned}$$

where both n and i are negative.

Eliminating as in the case where both n and i are positive, we get

$$J_{n+i}(x) = \bar{g}_{i-1}^{(n)}(x) J_{n-1}(x) - \bar{g}_{i-2}^{(n)}(x) J_n(x)$$

and readily obtain the relation

$$\bar{g}_i^{(n)}(x) = (-1)^i g_i^{(n)}(x).$$

Thus since the roots of $g_{i-1}^{(n)}(x)$ are all real, we have *the roots of $\bar{g}_i^{(n)}(x)$ are all real.* If n is an integer we have $J_{n+i}(x) = (-1)^{n+i} J_{-n-i}(x)$ and $J_n(x) = (-1)^n J_{-n}(x)$, so that there are $E\left(\frac{-i-1}{2}\right)$ of the intervals determined by the consecutive

* When $J_{n+\kappa}(x)$ and $J_n(x)$ have a common root it may be regarded as lying in either of the intervals abutting on the common root.

† Hurwitz shows by a different method, p. 255, l. c. Math. Ann., Bd. 33, that when $n > 0$ all the roots of $g_{i-1}^{(n)}(x)$ are real, but does not show that not more than *one* root of $g_{i-1}^{(n)}(x)$ can lie in the interval delimited by two consecutive roots of $J_n(x)$.

positive roots of $J_n(x)$ in which there is one root of $g_{i-1}^{(n)}(x)$ and no root of $J_{n+i}(x)$. When n is not integral it is only necessary to let n increase to the value $-E(-n)$. We know that before n began to increase there were at most $E\left(\frac{-i-1}{2}\right)$ vacant intervals to the right of the origin, and when n reaches the value $-E(-n)$ we have seen that there are exactly $E\left(\frac{-i-1}{2}\right)$ such vacant intervals. It is, moreover, clear that when n increased to the value $-E(-n)$, no intervals were gained since the large roots of $J_{n+i}(x)$ and $J_n(x)$ all move out by equal amounts. Thus: *in each of the intervals delimited by the consecutive positive roots of* $J_n(x)$, $n<0$, *lies one and but one root of* $J_{n+i}(x)$, $i<0$, *except in* $E\left(\frac{-i-1}{2}\right)$ *of these intervals in which no root of* $J_{n+i}(x)$ *lies.*

The method hitherto employed does not enable us to treat the question of the reality of the roots of $g_i^{(n)}(x)$ when n and i have different signs. Here, in general, as the method of Hurwitz* readily shows, imaginary roots present themselves. The analytic expression for $g_{i-1}^{(n)}(x)$ on page 201 shows, however, that if $i=2\varkappa+1>0$ and $n<0$; if $\varkappa<-n<\varkappa+1$, all the roots of $g_{i-1}^{(n)}(x)$ are imaginary $\left(\text{all the coefficients of powers of } \frac{2}{x} \text{ being of same sign}\right)$, so that *the positive roots of* $J_{n+i}(x)$ $(n<0$ *and* i *odd) and* $J_n(x)$ *separate each other, if* $\frac{i-1}{2}<-n<\frac{i+1}{2}$ *or when* i *is even when* $\frac{i-2}{2}<-n<\frac{i+2}{2}$.

§5.—*On the Complex and Pure Imaginary Roots of* $J_n(x)$.

Hurwitz, in his paper "Ueber die Nullstellen der Bessel'schen Function," which has already been referred to, first enumerated the imaginary roots of $J_n(x)$, when n is real,† and determined the regions of the complex plane in which these roots lie both when n is real and when n is pure imaginary.

The theorem on page 197 affords an easy means of solving the first of these questions, and it is to this that we shall next turn our attention.

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* Hurwitz, Math. Ann., Bd. 33, l. c.

†It has long been known that when $n \geqq -1$, all the roots of $J_n(x)$ are real; the proof follows at once by the method due to Poisson (Theorie de la Chaleur, p. 178) from the integral

$$\int_0^a \nu J_n(\mu_\kappa \nu)\, J_n(\mu_\iota \nu)\, d\nu = \frac{a}{\mu_\kappa^2-\mu_\iota^2}\left[\mu_\kappa J_n(\mu_\iota a)\, J_{n+1}(\mu_\kappa a) - \mu_\iota J_n(\mu_\kappa a)\, J_{n+1}(\mu_\iota a)\right].$$

The imaginary roots of

$$J_n(x) = \frac{1}{\Gamma(1+n)} \left(\frac{x}{2}\right)^n \left[1 - \frac{1}{1+n}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,(1+n)(2+n)}\left(\frac{x}{2}\right)^4 - \text{etc.}\right]$$

are the same as the imaginary roots of

$$f(x, n) = \frac{1}{\Gamma(1+n)} \left[1 - \frac{1}{1+n}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,(1+n)(2+n)}\left(\frac{x}{2}\right)^4 - \right]$$
$$= \frac{1}{\Gamma(1+n)}\,\phi(x, n),$$

where it is to be noted that $f(x, n)$, regarded as a function of the two complex arguments x and n, has the following important properties:

If S denote a piece of the finite region of the x-planes, and Z a piece of the finite region of the n-planes.

1°. For any given value of n in the region Z, $f(x, n)$ is a finite, continuous, and singly-valued analytic function of x throughout S.

2°. For any value of x in the region S, $f(x, n)$ is a finite, continuous, and singly-valued analytic function of n throughout Z.

To prove these properties of $f(x, n)$ it is only necessary to note that for values of n in the neighborhood of a real negative integer $-i$, we have

$$\frac{1}{\Gamma(1+n)} = (n+i)\,\omega_1(n+i),$$

where $\omega_1(n+i)$ is analytic at $n = -i$, while $\phi(x_1, n) = \frac{1}{n+i}\,\omega_2(x_1, n+i)$, where ω_2 is analytic at $n = -i$; so that, since the product of two analytic functions is analytic, $f(x, n)$ is an analytic function of either x or n in the regions in question.

Concerning functions of two arguments, which satisfy the conditions 1° and 2°, the following general theorem* holds:

If $f(x, n) = 0$, where n is a point within the region Z, has no roots on the boundary of the region S; the number† of roots of $f(x, n)$ within S will remain unchanged, if n be changed to $n + \Delta n$, where $|\Delta n| < \rho$ and where ρ is a sufficiently small positive quantity.

---

* See, for instance, Neumann's Abelsche Integrale, p. 141.

† A κ-fold root is, of course, to be counted as κ simple roots.

ose that n, starting with a value $-\frac{1}{2}$, in which case

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \cos x,$$

) are real, decrease through the value -1. All tl
towards the origin, and when n passes through tl
ach the origin when $n = -1$, disappear from the
fore, by the theorem just quoted, become conjugat
ay, when n continually decreasing passes through tl
become conjugate imaginary. When n decrease
, $2E(-\bar{n})$ roots become conjugate imaginary by dis
eals at the origin. Since $J_n(x)$* can have no multip
imaginary roots of $f(x, n)$ must arise at the origin
sy to show, however, by means of the asymptotic
ry roots can come in from the point ∞. In the r
ht of the axis of pure imaginaries, we have asympto

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \cos\left\{\frac{2n+1}{4}\pi - x\right\},$$

pure imaginaries,‡

$$J_n(x) = i^n \sqrt{\frac{i}{2\pi x}} e^{-xi}.$$

root of $J_n(x)$ so also is $-x_0$, we need only consi
ounded by the positive halves of the axes of reals a
the asymptotic values given above show that all tl
l.

ons on pages 197-8 show that when $-1 \overline{\gtrless} n \overline{\gtrless} -\frac{1}{2}$,
1. If then we let n, starting with a value a little
ll the roots of $f(x, n)$ will move away from the orig
ear from the axis of real, we have the theorem:

a negative integer, $J_n(x)$ has $2E(-n)$ conjugate im
infinite number of real roots.

---

e the same roots save at the origin.

Bessel Functions, p. 70, and Jordan, Cours d'Analyse, tom. 3, p.

68.

al value, the moduli of all the imaginary
e, when n is integral, we have $J_n(x)$
ots are real.
etrical with respect to the origin and the
ust be at least two pure imaginary roots.
e deduced by the considerations we have

as the independent variable, that when

imaginary roots, while if $E(-n)$ is even,

y questions relating to the roots of the
s of the continued fraction for J_n/J_{n+1}.
icable to functions defined by any homo-
f the second order; but as the variable
e theorems obtained become more special

e exponent-differences being

$$\lambda = \gamma - 1, \quad \mu =$$

ere it is supposed that α, β, and γ

By the change of dependent var

$$y = \bar{y}\, x^{-}$$

becomes

$$\frac{d^2\bar{y}}{dx^2} = \phi\,(\lambda, \mu, \nu, x)\,\bar{y}$$

$$= -\tfrac{1}{4}\,\frac{x^2(1-\mu^2)+}{}$$

When λ is positive, and in the p
s is the case, the solution correspo
inary hypergeometric series

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha}{1!}$$

ere, since α and β are interchange

$$\mu =$$

reover, since

$$\frac{\partial\phi\,(\lambda, \mu, \nu,}{\partial\mu}$$

see that ϕ decreases with the incr

By III′ we see that if $x_r = \phi_r(\mu)$
ween 0 and 1, x_r will *increase* as μ
the case $\lambda < 0$, and as yet I hav
roots of a solution corresponding

The next question that we shall
mall amount, $\Delta\mu$, λ and ν rema
ts of

$$y\,(x, \mu) =$$

ween 0 and 1 are slightly increas
root of $y\,(x, \mu - \Delta\mu)$ in each c
ned by the roots of $y\,(x, \mu)$ bet

* Using x_∞ to denote the root of

ch of these intervals.
ie explicit expression
ndependent solutions

$\beta + 1,\ 1 - x),\quad (2)$

$\overline{\nu}\Big)$,

$\overline{\nu}\Big)$

$,\ 1 + \lambda,\ x\Big).$

positive, and it can

ease; all the roots of
of the intervals deter-
en 0 and 1, there will

altered until the root
as μ decreases, disap-
ermine when the root
e.

x a little less than 1,

$+\nu\Big) = \Gamma(\beta)$. When,

next negative integer,
ved up to the point 1,
sses through the *next*
int 1 and we have the

theorem: *When* $\nu > 0$, $F(\alpha -$
each of the intervals x_1x_2, x_2x_3, ..
if $0 < \varkappa \leqq -\beta - E(-\beta)$ *the*
more generally: $F(\alpha - \varkappa, \beta +$
the intervals x_1x_2, $x_2n_3 \ldots . x_\omega 1$
$F(\alpha - \varkappa, \beta + \varkappa, \gamma, x)$ *lies, if*

$$\eta + i - 1$$

when $\eta = -\beta - E(-\beta)$.

When $\nu < 0$ (we assume th

$$F(\alpha, \beta, \gamma, x) = ($$

and since $F(\gamma - \beta, \gamma - \alpha, \gamma$
exponent differences λ, μ,
$F(\alpha - \varkappa, \beta + \varkappa, \gamma, x)$ *has one*
$x_2x_3 \ldots . x_\omega 1$, *except in* i *of*
$F(\alpha - \varkappa, \beta + \varkappa, \gamma, x)$ *if*

$$\eta -$$

where $\eta = \alpha - \gamma -$

In precisely the same way,

1°. $$\frac{\partial \phi}{\partial \lambda} = \frac{1}{2}$$

if $\lambda > 0$.

2°. $$\frac{\partial \phi}{\partial \nu} = \frac{1}{2}$$

if $\nu > 0$.

By 1° $F(\alpha + \varkappa, \beta + \varkappa, \gamma +$
intervals x_1x_2, ..., $x_\omega 1$, *delimi*
there will be no root of $F(\alpha + \varkappa,$

$$\eta + i -$$

when, as before, $\eta = -\beta - E$
$-E(\alpha - \gamma)$ and $\alpha - \gamma - \varkappa >$

* When $F(\alpha, \beta, \gamma, x)$ and $F(\alpha - \kappa, \beta +$
either of the intervals separated by the c

triction

$< 0.$

en $\nu > 0$, $F(\alpha + x, \ \beta + x, \ \gamma, \ x)$,

ıpe of the circular triangles on which the x-halfplane is mapped, has be ved more recently by Hurwitz* and Gegenbauer.† The solutions of Hurw 1 Gegenbauer both depend on the determination of a chain of contiguo pergeometric functions which can be used as a set of Sturmian functio ein's method, on the contrary, only makes use of the differential equation, b ile extremely elegant and interesting, does not lead to this result so direct the methods of Sturm which we have been employing.

On page 210 we saw that when $\lambda > 0$ and $\nu > 0$, one root in the interv was lost whenever β, through the decrease of μ, increased through a negati eger. When β, always increasing, reaches a value a little greater than ze $(1-\beta)$‡ roots will have been lost, and as the hypergeometric series then h its terms positive and can have no roots > 0, all the roots between 0 and ıst have been lost. Thus when $\lambda > 0$ *and* $\nu > 0$, *the hypergeometric ser* s $\bar{E}(1-\beta)$ *roots between* 0 *and* 1, and in the same way by the consideratic page 211 when $\lambda > 0$ *and* $\nu < 0$ *the hypergeometric series has* $\bar{E}(1-(\gamma-$ c *ots between* 0 *and* 1.

When $\lambda < 0$, we must take, not the hypergeometric series, but

$$y = x^{-\lambda} F\left(\frac{1-\lambda+\mu+\nu}{2},\ \frac{1-\lambda-\mu+\nu}{2},\ 1-\lambda,\ x\right),$$

ıich is then the solution corresponding to the larger exponent of the orig ıe hypergeometric series on the right which we may denote by $F(\alpha', \beta', \gamma',$ isfies a differential equation whose exponent differences are $-\lambda, \mu, \nu$, so th *the interval* 01 $F(\alpha', \beta', \gamma', x)$ *has* $\bar{E}(1-\beta')$ roots if $\nu > 0$ and $\bar{E}[1-(\gamma'-\alpha$ ots if $\nu < 0$.

All four results can be stated in the one formula

$$\chi = \bar{E}\left(\frac{\mu-|\lambda|-|\nu|+1}{2}\right).$$

When $\lambda < 0$, we determined not the number of roots of $F(\alpha, \beta, \gamma,$ tween 0 and 1, but the number of roots of a solution which, in general, ıearly independent of $F(\alpha, \beta, \gamma, x)$.

Theorem I′ of the introduction shows that in this case $F(\alpha, \beta, \gamma, x)$ h $=\chi$ or $\chi + 1$ roots between 0 and 1, the even or odd value of N being chos cording as $y_{x=1}$ is $>$ or <0. There are thus two cases to be considered.

* Math. Ann., Bd. 38. † Sitz. Bericht., Wien. Akad., Bd. 100, 2*a*.

‡ $\bar{E}(s)$ denotes the largest positive integer less than s, while $\bar{E}(s) = 0$ if $s \leq 1$.

Iere, according as b is $>$ or <0, i. e. according as

$$\Gamma\left(\frac{1+\lambda+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\lambda-\mu+\nu}{2}\right) \text{ is } > \text{ or } < 0,$$

$>$ or <0.

Iere, according as a is $>$ or <0, i. e. according as

$$\Gamma\left(\frac{1+\lambda-\mu-\nu}{2}\right)\Gamma\left(\frac{1+\lambda+\mu-\nu}{2}\right) \text{ is } > \text{ or } < 0,$$

), i. e. if $(1+\lambda+\mu+\nu)/2$ or $(1+\lambda-\mu+\nu)/2$ is zero or a $F(\alpha, \beta, \gamma, x)$ ceases to be linearly independent of $F(\alpha, \beta,$ $-x)$, which, when $\nu>0$, is the solution corresponding to the f $x=1$, so that by the theorem I′ of Sturm $F(\alpha, \beta, \gamma, x)$ has s between 0 and 1.

reason in 2°, when $a=0$, i. e. when $(1+\lambda-\mu-\nu)/2$ or is zero or a negative integer, the hypergeometric series has χ nd 1.

iced that in discussing the general problem of enumerating the $\gamma, x)$, we have been led naturally to consider the case where ve integer, and have in a certain sense generalized the results tes Rendus, tom. 100, and Hilbert, Crelle, Bd. 103, which refer

SITY, *June* 8, 1897.

ιtion of the Problem o
ransformation of a Bip

By Thomas Muir, LL. D.

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tite quadric be

$$\begin{array}{cccc|c} x & y & z \ldots\ldots & & \\ \hline a_1 & a_2 & a_3 \ldots\ldots & & x' \\ b_1 & b_2 & b_3 \ldots\ldots & & y' \\ c_1 & c_2 & c_3 \ldots\ldots & & z' \\ \ldots\ldots & & & & \ldots \end{array}$$

nd two matrices

$$\begin{pmatrix} {}_1 & \alpha_2 & \alpha_3 \ldots\ldots \\ {}_1 & \beta_2 & \beta_3 \ldots\ldots \\ {}_1 & \gamma_2 & \gamma_3 \ldots\ldots \\ \ldots\ldots & & \end{pmatrix}, \qquad \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \\ r_1 & r_2 \\ \ldots\ldots & \end{pmatrix}$$

orming the substitutions

$$, z, \ldots\ldots) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \ldots\ldots \\ \beta_1 & \beta_2 & \beta_3 \ldots\ldots \\ \gamma_1 & \gamma_2 & \gamma_3 \ldots\ldots \\ \ldots\ldots & & \end{pmatrix}($$

$$, z', \ldots\ldots) = \begin{pmatrix} m_1 & m_2 & m_3 \ldots\ldots \\ n_1 & n_2 & n_3 \ldots\ldots \\ r_1 & r_2 & r_3 \ldots\ldots \\ \ldots\ldots & & \end{pmatrix}($$

ξ'
η'
ζ'
$\cdot\ \cdot$

the given bipartite

$m_2\ m_3\ \cdots\cdot$
$\eta'\ \ \zeta'\ \ \cdots\cdot$

$n_2\ \ n_3\ \ \cdots\cdot$
$\eta'\ \ \zeta'\ \ \cdots\cdot$

$r_2\ \ r_3\ \ \cdots\cdot$
$\eta'\ \ \zeta'\ \ \cdots\cdot$

quadric by D, and

of the two substitutions by S_1
e written*

$$\frac{\xi \quad \eta \quad \zeta \,.}{S_1}$$

is by another property of bipar

$$\frac{\xi \quad \eta}{tr\ S_2 \,.}$$

t the problem is changed into
the equation

$$tr\ S_2 \,.$$

roblem is evidently indeterminat
atrical equation to be satisfied,
for obtaining them; the soluti
constants.

. It is immediately clear that
l one—by putting S_1 or S_2 equa
or S_1. Doing this and denotin
ative solutions

$$S_1 =$$

$$tr\ S_2 =$$

$$tr\ S_2 =$$

$$S_1 =$$

r D" is used, as by Cayley, for the 'tran
l-column exchange: "conj. D" would
conjugate' is already used in connection

e equation will be
natrix and D, and
us

, though in other

e following simple

$', \eta', \zeta' \ldots .)$

,

,

ant.

xisymmetric, equal

:, of this is axisym-
luct

ne unknown matrix
ılar unknowns, and
m is thus again an
s to be expected in

it follows therefore tl
other factors—the firs

and bearing in mind t
the law of row-and-col

and

The first and third fac

and our equation is

$(tr\,S + 1)\{$

The solution got from
third. Taking the re

$A -$

or $\{A -$

the solution of which

$A -$

Denoting by $-N$ su
involving as was desir

and ∴

and finally

7. We have thus

The transformatio

$(x, y, z, \ldots$

can be effected by the s

$(x, y, z$

except for the condition

he much simpler case of
form of a "rule," viz.
on which transforms the

$xy + \ldots\ldots$

Δ

$)$ when $r \neq s$
$) - \Delta,$

r the very simplest case.
$\ldots\ldots = 1$ and $f = g = h$

only even powers of the
$= -\mu$ is also a solution.

$4\}(S + 1),$

$A\}(S - 1),$
ianifest as the solution

9. Another point worthy of attention is the fact that the identity of

$$(tr\, S+1)\{A-A(S+1)^{-1}-(tr\, S+1)^{-1}A\}(S+1)$$

d

$$tr\, S.A.S-A$$

quite independent of the form of A, and that therefore whatever A may be, root of

$$A-A(S+1)^{-1}-(tr\, S+1)^{-1}A=0$$

a root of

$$tr\, S.A.S-A=0.$$

hen, however, we proceeded, towards the close of §6, to solve for the root of former equation, we introduced the condition $A=tr\, A$: consequently when is unconditioned* it cannot be expected that the root thus reached will be a t of the equation

$$tr\, S.A.S-A=0.$$

10. Returning now to the general equation with which we started, viz.

$$tr\, S_2.D.S_1=D,$$

us give S_1 a form analogous to the form $2(A+N)^{-1}A-1$ obtained for S §6, changing A, of course, into the more general D and, in order that we may ve the full number of arbitrary constants, putting a perfectly arbitrary matrix in place of the zero-axial skew matrix N. Our equation then becomes

$$tr\, S_2.D.\{2(D+M)^{-1}D-1\}=D$$
$$tr\, S_2.\{2D(D+M)^{-1}D-D\}=D$$
$$tr\, S_2.\{2D(D+M)^{-1}-1\}D=D,$$

d ∴

$$tr\, S_2=\{2D(D+M)^{-1}-1\}^{-1}.$$

11. In place therefore of the simple theorem of §5 we have the following as useful alternative:

The transformation of the bipartite quadric

$$(x,y,z,\ldots)(D)(x',y',z',\ldots) \text{ into } (\xi,\eta,\zeta,\ldots)(D)(\xi',\eta',\zeta',\ldots)$$

---

* The solution of $A-A(S+1)^{-1}-(tr\, S+1)^{-1}A=0$, when A is unconditioned, does not seem to easy. As we see readily from the complete equation $tr\, S.A.S=A$, there are then implied n^2 ations for the determination of n^2 unknowns.

or, as the saying is, into itself, can be effected by the substitutions

$$\begin{aligned}(x, y, z, \ldots\ldots) &= \quad (2\Delta^{-1}D - 1)(\xi, \eta, \zeta, \ldots\ldots),\\ (x', y', z', \ldots\ldots) &= tr\,(2D\Delta^{-1} - 1)(\xi', \eta', \zeta', \ldots\ldots),\end{aligned}$$

where $\Delta = D +$ *an arbitrary matrix and has a non-vanishing determinant.*

12. Bearing in mind the equation $tr\, S_2 . D . S_1 = D$, we see that the foregoing result rests finally on the identity

$$(2D\Delta^{-1} - 1)^{-1} D (2\Delta^{-1}D - 1) = D,$$

which is the same as

$$D(2\Delta^{-1}D - 1) = (2D\Delta^{-1} - 1)\,D.\text{*}$$

This latter however leads with equal naturalness to the identity

$$(2D\Delta^{-1} - 1)\,D\,(2\Delta^{-1}D - 1)^{-1} = D,$$

so that another form of the solution of the equation

$$tr\, S_2 . D . S_1 = D$$

is available, viz.

$$\left.\begin{aligned} S_1 &= (2\Delta^{-1}D - 1)^{-1},\\ S_2 &= tr\,(2D\Delta^{-1} - 1).\end{aligned}\right\}$$

13. The existence of an alternative form of solution is, of course, what might have been expected from the character of the function under discussion, which itself has two forms:

$$\begin{array}{cccc|c} x & y & z & \ldots\ldots & \\ \hline a_1 & a_2 & a_3 & \ldots\ldots & x' \\ b_1 & b_2 & b_3 & \ldots\ldots & y' \\ c_1 & c_2 & c_3 & \ldots\ldots & z' \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array} \quad \text{and} \quad \begin{array}{cccc|c} x' & y' & z' & \ldots\ldots & \\ \hline a_1 & b_1 & c_1 & \ldots\ldots & x \\ a_2 & b_2 & c_2 & \ldots\ldots & y \\ a_3 & b_3 & c_3 & \ldots\ldots & z \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

or $\qquad (x \;\; y \;\; z \ldots\ldots)(\;D\;)(x' \;\; y' \;\; z' \ldots\ldots),$

and $\qquad (x' \;\; y' \;\; z' \ldots\ldots)(tr\,D)(x \;\; y \;\; z \ldots\ldots).$

---

* The ultimate basis is, of course, the fact that

$$2D\Delta^{-1}D - D$$

is equal to either $\qquad D(2\Delta^{-1}D - 1) \;$ or $\; (2D\Delta^{-1} - 1)\,D.$

Consequently, when we obtained the solu

$$S_1 = 2\Delta^{-1}D -$$
$$S_2 = tr\,(2D\Delta^-$$

we might, by attending to the evidently the solution

$$S_2 = 2\ tr\ \Delta^{-1}\ tr$$
$$S_1 = tr\ (2\ tr\ D\ tr$$

And this latter will be found to be iden above; for—to take only the value for S_1

$$tr\ (2\ tr\ D\ tr\ \Delta^{-1} - 1)^{-1} = \{$$
$$= \{$$
$$= ($$

14. Further, however, it must be no the unlikeness—which is notable in both of S_1 and the value of S_2; and the presu either form must be suitable for both, is taking the form for $tr\ S_2$ in the first solut

$$(2D\Delta^{-1} - 1)^{-1} = \{(2D -$$
$$= \Delta\,(2D -$$
$$= \{2D -$$
$$= 2D\,(D -$$

which closely resembles the form for S_1 the form for S_1, we have

$$2\Delta^{-1}D - 1 = \Delta^{-1}\,(2$$
$$= \Delta^{-1}\,(D$$
$$= \{(D -$$
$$= \{(D -$$
$$= \{2D\,(D$$

which is quite similar to the form for tr

15. And this is not all, for, since

$$(D + M)^{-1}(D$$

and $\therefore$ $$(D + M)^{-1}D +$$

we see that each of the eight
in it is the same as to take its
on of §13, the values of S_1 and
n the first solution, it follows
by merely changing the sign
ly the second solution is not

licated forms of S_1 and S_2 is

ay, and $S_1 = S_2$, we have for

$-M)^{-1} - 1\}$,
$)^{-1}\, tr\, A$,
$M)^{-1}$,

18. In order that
append hereto a first ap
A supplementary list, t
is sufficiently bulky to

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On the Primitive Substitution Groups of Degree Sixteen.

By G. A. Miller.

The object of this paper is to determine all the primitive groups of degree 16 and to study them in regard to solvability. It will be seen that the number of these groups is considerably larger than it has been supposed to be.* The paper is divided into two sections. In the first we determine the groups by very simple methods and study them with respect to the given property. In the second we prove that it is impossible to construct a primitive group of degree 16 that is not given in the first section.

§1.

Determination of the groups.

It will be proved that all these groups, with the exception of the two that include the alternating group of this degree, contain a self-conjugate subgroup of order 16. As a self-conjugate subgroup of a primitive group is transitive, this must be regular. Since the entire group must transform each of its substitutions except identity into a system that generates it, it must be the Abelian group that contains 15 subgroups of order 2. We shall denote it by H and suppose that its substitutions are

1		$ai.bj.ck.dl.em.fn.go.hp$	I,
$ab.cd.ef.gh.ij.kl.mn.op$	B,	$aj.bi.cl.dk.en.fm.gp.ho$	J,
$ac.bd.eg.fh.ik.jl.mo.np$	C,	$ak.bl.ci.dj.eo.fp.gm.hn$	K,
$ad.bc.eh.fg.il.jk.mp.no$	D,	$al.bk.cj.di.ep.fo.gn.hm$	L,
$ae.bf.cg.dh.im.jn.ko.lp$	E,	$am.bn.co.dp.ei.fj.gk.hl$	M,
$af.be.ch.dg.in.jm.kp.lo$	F,	$an.bm.cp.do.ej.fi.gl.hk$	N,
$ag.bh.ce.df.io.jp.km.ln$	G,	$ao.bp.cm.dn.ek.fl.gi.hj$	O,
$ah.bg.cf.de.ip.jo.kn.lm$	H,	$ap.bo.cn.dm.el.fk.gj.hi$	P.

* Only 12 of these groups that do not contain the alternating group are given in Jordan's enumeration, Comptes rendus, vol. 75, p. 1757. We shall prove the existence of 20 such groups.

The group of isomorphisms of H* is evidently doubly transitive, of degree 15 and order $15.14.12.8 = 8! \div 2$. It is known that there is only one group that satisfies these conditions, and that it is simply isomorphic to the alternating group of degree 8 (K). To every subgroup of K corresponds a transitive group of degree 16 that contains H as self-conjugate subgroup. Its subgroup that includes all its substitutions which do not involve a given element is simply isomorphic to the corresponding subgroup of K. We shall confine our attention to the primitive groups.

The subgroup of the group of isomorphism that corresponds to a primitive group must transform each of the substitutions of H, excepting identity, into substitutions that generate H. It must therefore be of degree 15, and its order cannot be less than 5. To a subgroup of order 5 in K corresponds a group of order 80 that contains H as self-conjugate subgroup. This group (G_1) must be primitive, for it contains no self-conjugate subgroup besides H and 1, and there is no transitive group of degree 8 and order 80. It is solvable since its factors of composition are 2, 2, 2, 2, 5. It may be generated by H and

$$boejc \,.\, dpknl \,.\, fhgim \,.†$$

From the preceding paragraph it follows that to every subgroup of K whose order is divisible by 5 there corresponds a primitive group of degree 16 that contains H as self-conjugate subgroup. If we include K, there are 16 such subgroups, viz. $(abcde)$ cyc.,‡ $(abcde)_{10}$, $(abcde)$ cyc. (fgh) cyc., $\{(abcde)_{20}(fg)\}$ pos., $(abcde)_{10}(fgh)$ cyc., $(abcde)$ pos., $(abcdef)_{60}$, $\{(abcde)_{20}(fgh)$ all$\}$ pos., $\{(abcde)$ all $(fg)\}$ pos., $\{(abcdef)_{120}(gh)\}$ pos., $(abcde)$ pos. (fgh) cyc., $(abcdef)$ pos., $\{(abcde)$ all (fgh) all$\}$ pos., $\{(abcdef)$ all $(gh\}$ pos., $(abcdefg)$ pos., $(abcdefgh)$ pos. As all the subgroups of K that are similar to any one of these are conjugate, there can be no more than 16 primitive groups of degree 16 that contain H as self-conjugate subgroup and whose orders are divisible by 5. We proceed to prove that no two of these 16 primitive groups are simply isomorphic and to find their factors of composition and generating substitutions.

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* Hölder, Mathematische Annalen, vol. 48, p. 314.

† The substitutions that are to be added to H to generate the required groups permute the substitutions of H in exactly the same manner as their own elements; i. e. they have been selected in such a way they they are the same as the corresponding ones in capital letters.

‡ Cayley, Quarterly Journal of Mathematics, vol. 25, p. 71.

The group of order 160 (G_2) may be generated by G_1 and

$$bj\,.\,dl\,.\,eo\,.\,fh\,.\,gm\,.\,np.$$

Since it contains G_1 as maximal self-conjugate subgroup, its factors of composition may be found by adding 2 to those of G_1. Hence it is solvable. The group of order 240 (G_3) may be generated by G_1 and

$$bmn\,.\,cik\,.\,deh\,.\,flo\,.\,gpj.$$

As it contains G_1 as maximal self-conjugate subgroup, its factors of composition may be obtained by adding 3 to those of G_1. It evidently contains 16 conjugate subgroups of order 15. The groups of orders 320 and 480 (G_4, G_5) may be generated, respectively, by G_2 and

$$bejo\,.\,dglm\,.\,fnhp\,.\,ik,\quad bmn\,.\,cik\,.\,deh\,.\,flo\,.\,gpj.$$

Since G_2 is a maximal subgroup of each of these groups, their factors of composition may be obtained by adding 2, 3 respectively to those of G_2. Hence they are solvable. G_5 contains G_3 as self-conjugate subgroup.

Since K contains 3 subgroups of order 60, there are 3 primitive groups of degree 16 and order 960 that contain H as self-conjugate subgroup. Only one of these (G_6) is solvable. It may be generated by G_5 and

$$bejo\,.\,dglm\,.\,fnhp\,.\,ik.$$

It is solvable since its factors of composition may be obtained by adding 2 to those of G_5. We have now found the 6 solvable primitive groups of degree 16 whose orders are divisible by 5. The remaining 12 groups whose orders satisfy this condition are insolvable.

The remaining two groups of order 960 correspond to

$$(abcdef)_{60},\quad (abcde)\text{ pos.}$$

in K. Hence each of them contains only two self-conjugate subgroups, viz. H and 1. Their factors of composition are 2, 2, 2, 2, 60. The former (G_7) may be generated by G_2 and

$$bpo\,.\,cnd\,.\,ekl\,.\,ghi.$$

The latter (G_8) may be generated by H and

$$boifc\,.\,dpgnh\,.\,emjlk,\quad bop\,.\,cge\,.\,dil\,.\,fmj\,.\,hkn.$$

Since the substitutions of order 3 in G_7 permute only 12 of the substitutions of H while those of G_8 permute 15, these two groups are not simply isomorphic.

The two primitive groups of order 1920 (G_9, G_{10}) correspond to

$$\{(abcdef)_{120}\,(gh)\}\text{ pos.},\quad \{(abcde)\text{ all }(fg)\}\text{ pos.}$$

in K. Hence each of them contains 3 self-conjugate subgroups, viz. H, 1 and the one of order 960. Their factors of composition are obtained by adding 2 to those of the preceding two groups. They may be generated by adding

$$bdjl\,.\,ck\,.\,epon\,.\,fmhg,\quad bpch\,.\,di\,.\,ejmk\,.\,fgon$$

to G_7 and G_8 respectively. They cannot be simply isomorphic since their self-conjugate subgroups of order 960 do not have this property.

The group of order 2880 (G_{11}) may be generated by G_8 and

$$bmn\,.\,dil\,.\,ceg\,.\,hoj\,.\,fkp.$$

Its factors of composition may evidently be obtained by adding 3 to those of G_8. The two groups of order 5760 (G_{12}, G_{13}) correspond to

$$\{(abcde)\text{ all }(fgh)\text{ all}\}\text{ pos.},\quad (abcdef)\text{ pos.}$$

Since the former contains G_{11} as self-conjugate subgroup, its factors of composition may be obtained by adding 2 to those of G_{11}. This is the last of the six primitive groups of degree 16 whose solution depends only upon that of the alternating group of degree 5. It may be generated by G_{11} and

$$bpch\,.\,di\,.\,ejmk\,.\,fgon.$$

The factors of composition of G_{13} are evidently 2, 2, 2, 2, 360. It may be generated by G_7 and

$$bop\,.\,cge\,.\,dil\,.\,fmj\,.\,hkn.$$

For

$$bnloj\,.\,cpekd\,.\,fhimg \times bj\,.\,dl\,.\,eo\,.\,fh\,.\,gm\,.\,np = bpo\,.\,cnd\,.\,ekl\,.\,ghi,$$
$$bpo\,.\,ceg\,.\,dli\,.\,fjm\,.\,hnk \times bpo\,.\,cnd\,.\,ekl\,.\,ghi \;= bop\,.\,cki\,.\,deh\,.\,fjm\,.\,gnl.$$

Hence the given generator corresponds to a substitution in the same elements in K as G_7. It clearly corresponds to a substitution of degree 3.

The group of order 11520 (G_{14}) contains G_{13} as self-conjugate subgroup. Its factors of composition may therefore be obtained by adding 2 to those of G_{13}. It is generated by G_{13} and

$$bdjl\,.\,ck\,.\,epon\,.\,fmhg.$$

G_{13} and G_{14} are the only two primitive groups of degree 16 whose solution depends only upon that of the alternating group of degree 6 but not upon any one of its subgroups.

The group (G_{15}) which corresponds to the alternating group of degree 7 in K may be generated by H and

$$bioph jg \,.\, clnmekf, \quad bcd \,.\, emi \,.\, fol \,.\, gpj \,.\, hnk.$$

For the former of these generators is transformed into its square by

$$eln \,.\, fkm \,.\, gjp \,.\, hio.$$

Since the latter is commutative to this, it must correspond to a substitution of degree 3 in K whose elements are included in the substitution of degree 7 to which the former generator corresponds. The factors of composition of G_{15} are 2, 2, 2, 2, 2520.

The largest group that contains H as self-conjugate subgroup (G_{16}) may be generated by G_{15} and

$$bd \,.\, fh \,.\, jl \,.\, np.$$

For the alternating group of degree 8 contains only two types of substitutions of order 2. We have seen that the substitutions which correspond to those of the type $ab \,.\, cd$ permute 12 substitutions of H. The given generator must therefore correspond to a substitution of degree 8 in K. The factors of composition of G_{16} are 2, 2, 2, 2, 20160.

We have now considered all the possible primitive groups of degree 16 that contain H as a self-conjugate subgroup and whose order is divisible by 5. We have seen that 6 of these 16 groups are solvable, 6 others depend upon the solution of the alternating group of degree 6, while the remaining four depend upon the solution of the alternating groups of degrees 6, 7 and 8. There are 4 additional primitive groups that contain H as a self-conjugate subgroup. We proceed to consider these.

Since the groups

$(ae \,.\, bf \,.\, cg \,.\, dh)(abc)$ cyc. (efg) cyc.,* $(ae \,.\, bf \,.\, cg \,.\, dh)\{(abc)$ all (efg) all$\}$ pos.,
$(afbe \,.\, cg \,.\, dh)\{(abc)$ all (efg) all$\}$ pos., $(ae \,.\, bf \,.\, cg \,.\, dh)(abc)$ all (efg) all

are maximal subgroups that do not contain any self-conjugate subgroup besides

* Cayley, loc. cit.

identity of the four groups, in order,

$(ae.bf.cg.dh)(abcd)$ pos. $(efgh)$ pos., $(ae.bf.cg.dh)\{(abcd)$ all $(efgh)$ all$\}$pos.,
$(afbe.cg.dh)\{(abcd)$ all $(efgh)$ all$\}$ pos., $(ae.bf.cg.dh)(abcd)$ all $(efgh)$ all,

each of the latter four groups is simply isomorphic to a primitive group of degree 16.* As each of these groups is evidently solvable, its factors of composition are the same as the prime factors of its order.

The first (G_{17}) may be generated by H and

$$bjofpm.ceg.dnihlk, \quad bpo.cnd.ekl.ghi.$$

Hence all its self-conjugate subgroups, except identity, contain H. From this it follows independently that it is primitive. G_{18} may be generated by G_{17} and

$$bmpjof.cki.dghlen.$$

It contains G_{17} as self-conjugate subgroup. It is of order 576 while G_{17} is of order 288. G_{19} is of the same order as G_{18}. It may be generated by H and

$$bpo.cnd.ekl.ghi, \quad bm.cide.fojp.gklh.$$

G_{20} contains G_{17}, G_{18}, G_{19} as self-conjugate subgroups. It is of order 1152 and may be generated by G_{19} and

$$bjofpm.ceg.dnihlk.$$

We have now found the 20 primitive groups of degree 16 that do not contain the alternating group of this degree, and have seen that 10 of them are solvable while the remaining 10 are insolvable. Since the given generating substitutions, excluding H, do not contain a, they generate a subgroup G_1' whose order is obtained by dividing the order of the group by 16. If G_1' is α times transitive, the corresponding group is $\alpha+1$ times transitive. The class of G_1' is the same as that of the corresponding group, etc. The G_1' of G_{17} is the group of isomorphisms of H. By adding the alternating and the symmetric group to the preceding we obtain the 22 primitive groups of degree 16. The last two are evidently unsolvable. Their factors of composition are respectively $16!\div 2$; 2, $16!\div 2$.

§2.

Proof that there are no other primitive groups of degree sixteen.

As we have examined all the possible groups that contain H as a self-conjugate subgroup and whose orders are divisible by 5, we do not need to consider,

* Dyck, Mathematische Annalen, vol. 22, p. 102.

in what follows, the groups which satisfy these two conditions. We shall consider the simply transitive and the multiply transitive groups separately, beginning with the former. The group in question will generally be represented by G and its subgroup which contains all its substitutions that do not contain a given element by G'.

A.—*Simply transitive groups.*

We shall make frequent use of the following theorems:

Theorem I. G' cannot contain a transitive subgroup.

Theorem II. All the prime numbers which divide the order of one of the transitive constituents of G', divide the order of every other constituent.

Theorem III. If a transitive constituent of G' is of a prime degree, all its other transitive constituents are of an equal or a larger degree.

Theorem IV. If p^α is the highest power of a prime number that is contained in the order of G, the subgroups of order p^α are transformed by the substitutions of G according to a transitive group whose order is divisible by p^β but not by $p^{\beta+1}$. G contains a self-conjugate subgroup of order $p^{\alpha-\beta}$.

Theorem V. If G' contains a self-conjugate subgroup (H) of degree $n-\alpha$, n being the degree of G, H' must be the transform, with respect to substitutions of G, of $\alpha-1$ other subgroups of G' ($H'_1, H'_2, \ldots, H'_{\alpha-1}$). The substitutions of G that transform H'_β, ($\beta=1, 2, \ldots, \alpha-1$), into H' transform also all the substitutions of G' that are commutative to H'_β into substitutions of G'.

Theorem VI. G' transforms $H'_1, H'_2, \ldots, H'_{\alpha-1}$ according to the elements in one of its constituents of degree $\alpha-1$, and no two of these subgroups can have all their elements in common, nor can any of them contain all the elements of H'.

Theorem VII. The group generated by $H'_1, H'_2, \ldots, H'_{\alpha-1}$ is of degree $n-1$, and G' is a maximal subgroup of G.

Theorem VIII. Every self-conjugate subgroup of a primitive group is transitive.

G' contains a transitive constituent of degree 12.

The other constituent is the symmetric group of degree 3. To 1 in this group must correspond an intransitive subgroup of the constituent of degree 12 (H'). The systems of intransitivity of H' are systems of non-primitivity of the constituent of degree 12. These systems could not be of degree 2 since they would have to be transformed according to a regular group, and H' would then contain all the substitutions of G' whose degree <13.

The given systems could not be of degree 3 since they would have to be transformed according to a transitive group whose order is divisible by 4. As the degree of the systems of H' could evidently not exceed that of a constituent of G', it is impossible to construct a primitive group of degree 16 in which G' contains a transitive constituent of degree 12. From one of the given theorems it follows directly that G' could not contain a transitive constituent of degree 11.

G' contains a transitive constituent of degree 10.

The constituent of degree 5 must clearly be transitive, and its order must be the same as the order of G'; i. e. G' must be obtained by establishing a simple isomorphism between a transitive group of degree 10 and one of degree 5. When the order of G' is 5, 10, or 20, the subgroups of order 2^α in G must be transformed by its substitutions according to a transitive group of degree 5. Hence there must be a self-conjugate subgroup of order 2^β, $\beta > 0$, in G, according to the given theorem. Since the substitutions of order 5 are of degree 15, we have the congruence

$$2^\beta \equiv 1. \qquad \text{mod. } 5.$$

As $\alpha < 8$, $\beta = 4$. We have considered all such groups.

When G' is of order 60 or 120, G must transform its subgroups of order 2^α according to some transitive group of degree 5 or 15. In the former case G contains a self-conjugate subgroup of order 16, as we have just proved. In the latter case the corresponding group of degree 15 must be non-primitive, since the orders of the primitive groups are divisible by 9. Since such a non-primitive group could contain no substitution besides identity that leaves all its systems unchanged, it must be simply isomorphic to a transitive group of degree 5. Hence the preceding proof applies also to this case.

G' contains a transitive constituent of degree 9.

Since the order of G' cannot be divisible by 5, the constituent of degree 6 must be either non-primitive or intransitive. If the order of G' would exceed that of the constituent of degree 6, the constituent of degree 9 would be non-primitive and G' would contain an H' of order 3^α. As the substitutions of order 3 and degree < 9 in the constituent of degree 9 are commutative, those of order 3 in the constituent of degree 6 could not be commutative, i. e. this constituent would be transitive and it would contain 4 conjugate subgroups of

order 3. As this is clearly impossible, the order of G' is the same as that of the constituent of degree 6.

The order of G' cannot exceed that of the constituent of degree 9, since the order of the quotient group of the constituent of degree 6 with respect to a suitable self-conjugate subgroup would not be divisible by 9. Hence G' must be composed of two simply isomorphic groups whose degrees are 9 and 6. The latter must be transitive, for if it were intransitive all its subgroups of order 3 and degree < 15 would be self-conjugate.

From the preceding it follows that G' must be of order 18, 36, or 72. G must therefore contain 8 conjugate subgroups of order 3 and degree 12. As none of its substitutions besides identity can transform each of these subgroups into itself, it must be simply isomorphic to a transitive group of degree 8 and order 288, 576, or 1152. There are only 4 such transitive groups. We have seen that each of them contains one maximal subgroup that does not include any self-conjugate subgroup besides identity and whose order is obtained by dividing the order of the group by 16. In other words, we have seen that each of these groups is simply isomorphic to one primitive group of degree 16. As each of the given groups of degree 8 contains only one set of conjugate subgroups of the required type, each of them is simply isomorphic to only one primitive group of degree 16.

G' contains a transitive constituent of degree 8.

It is clear that the constituent of degree 7 must be transitive. Since the two transitive constituents of G' would be primitive, they would have to be simply isomorphic. Hence G' would be of order 168. Its substitutions of order 2 would be of degree 12. The 6 systems of such a substitution are transformed according to $\{(abcd)_4(ef)\}$ dim. by the substitutions of G'. The substitutions of G would have to transform these systems according to a transitive group of degree 6 and order 16. This is clearly impossible.

If G' would contain a transitive constituent of degree 7 the other would have to be transitive and of degree 8. We have just seen that this is impossible. We have now considered all the cases when G' contains a constituent whose degree exceeds 6. The remaining cases do not lead to any additional group, and are so simple that it seems unnecessary to consider them here.

B.—*Multiply transitive groups.*

We shall begin with the cases when G' is non-primitive and contains an intransitive self-conjugate subgroup (H_1). If H_1 contains 3 systems of intransitivity it must be composed of three simply isomorphic transitive groups of degree 5, since its order cannot be divisible by 25. If its order is 5, 10, or 20 it must contain a self-conjugate subgroup of order 16, as has been proved above for a similar case. Its order could not be 60 or 120, since the number of its subgroups of order 3 and degree 9 would not be divisible by 7.

If H_1 contains 5 systems of intransitivity its subgroup of order 3^a must satisfy one of the two congruences

$$3^a \equiv 1 \text{ mod. } 5, \qquad 3^a \equiv 3 \text{ mod. } 5,$$

since a substitution of order 5 in H_1 could not transform two of its subgroups of order 3 into themselves. Hence $\alpha = 0$, 1, or 4. If α were 4, H_1 would contain 5 or 10 subgroups of order 3 and degree 6, for only 20 such subgroups are found in the group of order 3^5, and the two of the same degree could not occur in H_1. In G each of these subgroups would have to be transformed into itself by a subgroup whose order is divisible by 5. This is clearly impossible. Hence $\alpha = 0$ or 1.

We have seen that G contains a self-conjugate subgroup of order 16 when $\alpha = 0$ or when $\alpha = 1$, and the systems of intransitivity of H_1 are permuted by G' according to the metacyclic group or one of its subgroups. When H_1 is of order 3 and G' transforms its systems according to the alternating group of degree 5, its 3 substitutions that correspond to a substitution of order 3 in this alternating group must be of order 3, two of them must be of degree 12 and the third of degree 15. All the substitutions of G' must therefore be commutative with each substitution of H_1. Hence all its subgroups of order 12 that do not include any self-conjugate subgroup are conjugate in two sets. Since one of these two simply isomorphic transitive groups* of degree 16 contains substitutions of order 3 and degree 9, we need to consider only one G'. We may suppose that it is generated by

$$adgjmbehlkncfilo, \quad adh\,.\,bei\,.\,cfg\,.\,mon.$$

The substitutions of G that transform the latter of these generators into itself must permute its systems according to the alternating group of degree 4,

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* Jordan, "Traité des Substitutions," p. 272.

since those of G' permute them according to the alternating group of degree 3. Hence and from the fact that $bc.dh.eg.if.kl.mn$ transforms the given G into itself, we may suppose that G contains one of the following substitutions:

$$\left.\begin{matrix} amdohn \\ andmho \\ aodnhm \end{matrix}\right\} \left\{\begin{matrix} bcefig \\ bgecif \\ bfegic \end{matrix}\right\} \left\{\begin{matrix} jp.kl \\ jk.pl \end{matrix}\right. .$$

By trial we find that only one of these 18 substitutions, viz. $amdohn.bcefig.jp.kl$, generates a group whose substitutions that do not contain p are contained in G'. The cube of this last generating substitution and its transforms form the 15 substitutions differing from identity of a self-conjugate subgroup of order 16. Hence the group which contains the G' that transforms the systems of H_1 according to the symmetric group of degree 5 must also contain a self-conjugate subgroup of order 16.

There could be no primitive group containing the H_1 of order 6 since such a group would have to contain one of the two preceding as self-conjugate subgroup. This is impossible, since the substitutions of degree 12 and order 3 in G' could not transform the negative substitutions of H_1 among themselves. Hence it remains only to consider the cases when G' is a primitive group of degree 15.

There are only 4 such primitive groups that do not include the alternating group, viz. those which are simply isomorphic to the symmetric group of degree 6 and the alternating groups of degrees 6, 7, 8. Each of these groups contains substitutions of degree 12 and order 3. It will not be difficult to find substitutions of G that are not contained in G' and that are commutative to such a substitution. We shall pursue this method to find all the possible groups and then prove that each of them contains a self-conjugate subgroup of order 16.

The G' of order 360 may be generated by

$$abc.efg.ijm.kln,\quad ahcf.bedg.imkn.jl,\quad adjkl.bmnci.eohfg.$$

Since the first of these generators is transformed into itself by 9 substitutions of G' which permute its first three systems cyclically, and all the substitutions of this type are conjugate, G must contain a subgroup of order 36 that transforms it into itself and permutes its systems according to the alternating group of degree 4. Hence it must contain one of the following substitutions:

$$\left.\begin{matrix} aebfcg \\ afbgce \\ agbecf \end{matrix}\right\} \left\{\begin{matrix} ikjlmn \\ iljnmk \\ injkml \end{matrix}\right\} \left\{\begin{matrix} dh.op \\ do.hp \\ dp.ho \end{matrix}\right.$$

As the required substitution must be transformed into its 5th power by $af.be.cg.dh.ij.kl$, it must be one of the three used as factors in the following products:

$$aebfcg.injkml.dh.op \times ahcf.bedg.imkn.jl = adcb.gh.jnlm.op,$$
$$afbgce.injkml.dh.op \times ahcf.bedg.imkn.jl = cd.ehgf.jnlm.op,$$
$$agbecf.injkml.dh.op \times ahcf.bedg.imkn.jl = abdc.efhg.jnlm.op.$$

From these products it follows that only the last can occur in G since G' is of class 12. The transforms of its cube generate a self-conjugate subgroup of order 16.

The G' of order 720 may be generated by the preceding G' and $ae.bf.cg.dh$. Since the squares of the first two of the given three products are not contained in its subgroup of order 48 generated by the first two given generators of the preceding G' and the one just given, there can be only one G that contains this G'. It evidently contains the same self-conjugate subgroup of order 16 as the preceding G. These two groups are doubly transitive. As the remaining two primitive groups of degree 15 that do not contain the alternating group are doubly transitive, the corresponding groups of degree 16 will be triply transitive.

The G' of order 2520 may be generated by

$$aem.bfn.gjk.hil,\quad cek.dfl.gin.hjm,\quad akodn.bfmij.cgehl.$$

All the substitutions of G' that are similar to the first of these generators are conjugate in G' and each is transformed into itself by 9 substitutions which permute 3 of its systems according to the alternating group. Hence G must contain one of the following substitutions:

$$\left.\begin{matrix} abefmn \\ afnmb \\ anebmf \end{matrix}\right\} \left\{\begin{matrix} ghjikl \\ gijlkh \\ gljhki \end{matrix}\right\} \left\{\begin{matrix} cd.op \\ co.dp \\ cp.do \end{matrix}\right.$$

Since the required substitution must be transformed into its 5th power by the last of the three generators given above, it must be one of the three employed in forming the following products:

$$anebmf.ghjikl.cd.op \times akodn.bfmij.cgehl = biopdglefkcnh,$$
$$anebmf.gijlkh.cd.op \times akodn.bfmij.cgehl = bi.cnhefklopdgj,$$
$$anebmf.gljkhi.cd.op \times akodn.bfmij.cgehl = biefkjl.cnhopdg.$$

From these products it follows that only the last can occur in G. The transforms of its cube generate a self-conjugate subgroup of order 16.

The G' of order 20160 may be generated by

$$ab\,.\,cd\,.\,ef\,.\,gh,\quad aem\,.\,bfn\,.\,gjk\,.\,hil,\quad akcgemj\,.\,bldhfni,\quad aocgjmk\,.\,bdnflih.$$

As all the substitutions of G' that are similar to the second of these generators are conjugate, this G must also contain one of the 27 substitutions given in the preceding case. Since the required substitution must be transformed into its 5^{th} power by $em\,.\,fn\,.\,gk\,.\,hl$, which is contained in G', it must be one of the three used as factor in the following equations:

$$anebmf\,.\,gljhki\,.\,cd\,.\,op \times akcgemj\,.\,bldhfni = aiel\,.\,bjfk\,.\,ch\,.\,dg\,.\,mn\,.\,op,$$
$$anebmf\,.\,gljhki\,.\,co\,.\,dp \times akcgemj\,.\,bldhfni = aiel\,.\,bjfk\,.\,cogdph\,.\,mn,$$
$$anebmf\,.\,gljhki\,.\,cp\,.\,do \times akcgemj\,.\,bldhfni = aiel\,.\,bjfk\,.\,cpgdoh\,.\,mn.$$

Hence only the first of these three can occur in a G. The transforms of its cube generate a self-conjugate subgroup of order 16. We have now considered all the possible primitive groups of degree 16 that do not contain the alternating group, and have found no group that is not contained in the enumeration of the first section. It may be observed that the substitutions upon which our arguments have been based may be selected in many different ways. As suitable substitutions can readily be found by means of the given generating substitutions, it did not seem necessary to indicate in every case how the particular one that has been employed has been obtained.

It is well known that a solvable primitive group must be of degree p^n, p being a prime number.* The following table gives the enumeration of all these groups whose degree is less than 27:

Degree,	3	4	5	7	8	9	11	13	16	17	19	23
Number of groups,	2	2	3	4	2	7	4	6	10	5	6	4

PARIS, *June*, 1897.

* Cf. Jordan, "Traité des Substitutions," p. 398.

Point Transformations in Elliptic Coordinates of Circles having Double Contact with a Conic.

BY E. O. LOVETT.

There are two systems of circles having double contact with a conic, the chords of contact of each system being parallel to one of the axes of the curve.

If
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
is the conic, these two systems are
$$x^2+y^2-2e^2x_1x+e^2x_1^2-b^2=0, \tag{1}$$
$$x^2+y^2+\frac{2c^2y_1}{b^2}y-\frac{c^2}{b^2}y_1^2-a^2=0, \tag{2}$$
corresponding respectively to the chords of contact
$$x-x_1=0, \quad y-y_1=0.$$

The equation (1) may be written in the form
$$x^2+y^2-2cx\cos\theta+c^2\cos^2\theta-b^2\sin^2\theta=0. \tag{3}$$
This equation is transformed into its equivalent in elliptic coordinates
$$\mu^2+\nu^2-2\mu\nu\cos\theta-a^2\sin^2\theta=0, \tag{4}$$
by assuming
$$\left.\begin{aligned} x^2+y^2&=\mu^2+\nu^2-c^2,\\ cx&=\mu\nu.\end{aligned}\right\} \tag{5}$$

The equation (4) may be replaced by the relation
$$\cos^{-1}\frac{\mu}{a}\pm\cos^{-1}\frac{\nu}{a}=\theta, \tag{6}$$
whence, by differentiation,
$$\frac{d\mu}{\sqrt{a^2-\mu^2}}\pm\frac{d\nu}{\sqrt{a^2-\nu^2}}=0, \tag{7}$$

the differential equation of the system of circles (1). Similarly we find that

$$\frac{\mu d\mu}{\sqrt{(\mu^2-a^2)(\mu^2-c^2)}} \pm \frac{\nu d\nu}{\sqrt{(a^2-\nu^2)(c^2-\nu^2)}} = 0 \tag{8}$$

is the differential equation of the second system of circles (2).

It is proposed now to find the general point transformations in the elliptic coordinates μ, ν which leave these families of circles invariant.

Let the general infinitesimal transformation of the group which leaves the first system of circles invariant be

$$Uf \equiv \xi(\mu, \nu)\frac{\partial f}{\partial \mu} + \eta(\mu, \nu)\frac{\partial f}{\partial \nu}. \tag{9}$$

Putting $\frac{d\nu}{d\mu} \equiv \rho$, the general infinitesimal transformation of the first extension of the original group is

$$U'f \equiv \xi(\mu, \nu)\frac{\partial f}{\partial \mu} + \eta(\mu, \nu)\frac{\partial f}{\partial \nu} + \chi(\mu, \nu, \rho)\frac{\partial f}{\partial \rho}, \tag{10}$$

where $$\chi(\mu, \nu, \rho) \equiv \eta_\mu + (\eta_\nu - \xi_\mu)\rho - \xi_\nu\rho^2. \tag{11}$$

By this transformation a function $\Omega(\mu, \nu, \rho)$ receives the increment

$$\delta\Omega \equiv U'\Omega\delta t \equiv \frac{\partial \Omega}{\partial \mu}\delta\mu + \frac{\partial \Omega}{\partial \nu}\delta\nu + \frac{\partial \Omega}{\partial \rho}\delta\rho, \tag{12}$$

where $$\delta\mu = \xi(\mu, \nu)\,\delta t, \quad \delta\nu = \eta(\mu, \nu)\,\delta t, \quad \delta\rho = \chi(\mu, \nu, \rho)\,\delta t. \tag{13}$$

If the function Ω is to be invariant, then $\delta\Omega$ is zero.

The function Ω for the case in hand is given by the equation (7), which may be written

$$\Omega(\mu, \nu, \rho) \equiv \sqrt{a^2-\nu^2} \pm \rho\sqrt{a^2-\mu^2} = 0. \tag{14}$$

The total variation of Ω becomes

$$\delta\Omega \equiv \mu\rho^2\delta\mu - \nu\delta\nu \pm \sqrt{(a^2-\mu^2)(a^2-\nu^2)}\,\delta\rho = 0, \tag{15}$$

which by the relations (13) assumes the form

$$\{\mu\xi \mp \sqrt{(a^2-\mu^2)(a^2-\nu^2)}\}\rho^2 \pm (\eta_\nu - \xi_\mu)\rho - \nu\eta \pm \eta_\mu\sqrt{(a^2-\mu^2)(a^2-\nu^2)} = 0, \tag{16}$$

which must be true for all values of ρ; accordingly we find the following equations of condition for the functions $\xi(\mu, \nu)$ and $\eta(\mu, \nu)$:

$$\frac{\mu\xi}{\xi_\nu} = \frac{\nu\eta}{\eta_\mu} = \pm\sqrt{(a^2-\mu^2)(a^2-\nu^2)}, \tag{17}$$

$$\xi_\mu = \eta_\nu. \tag{18}$$

The logarithmic integration of equations (17) gives

$$\left.\begin{aligned}\xi &= e^{\pm\Phi(\mu)\sin^{-1}\frac{\nu}{a}},\\ \eta &= e^{\pm\Psi(\nu)\sin^{-1}\frac{\mu}{a}},\end{aligned}\right\} \tag{19}$$

in which

$$\Phi(\mu)=\frac{\mu\phi(\mu)}{\sqrt{a^2-\mu^2}},\quad \Psi(\nu)=\frac{\nu\psi(\nu)}{\sqrt{a^2-\nu^2}}, \tag{20}$$

$\phi(\mu)$ and $\psi(\nu)$ being arbitrary functions. The equation (18) leads to the determination of the forms of Φ and Ψ, for, subjecting (19) to the condition (18), there results

$$\frac{\pm\Phi'(\mu)\,e^{\pm\Phi(\mu)}}{\sin^{-1}\dfrac{\mu}{a}\,e^{\sin^{-1}\frac{\mu}{a}}}\equiv\frac{\pm\Psi'(\nu)\,e^{\pm\Psi(\nu)}}{\sin^{-1}\dfrac{\nu}{a}\,e^{\sin^{-1}\frac{\nu}{a}}}=m\,; \tag{21}$$

whence, integrating by parts,

$$\Phi(\mu)=\pm\sin^{-1}\frac{\mu}{a}\pm\log\left\{\frac{m}{2}\left[(\mu+\sqrt{a^2-\mu^2})\sin^{-1}\frac{\mu}{a}-\mu\right]\right\},\ \Psi(\nu)=\Phi(\nu). \tag{22}$$

Accordingly the infinitesimal transformation sought has the form

$$Uf\equiv e^{\pm F(\mu,\nu)}\frac{\partial f}{\partial\mu}+e^{\pm F(\nu,\mu)}\frac{\partial f}{\partial\nu}, \tag{23}$$

where $\pm F(\mu,\nu)=\pm\Phi(\mu)\sin^{-1}\dfrac{\nu}{a}$.

In like manner we may find the infinitesimal point-transformations which leave invariant, respectively, the system of circles (8) and their orthogonal trajectories,

$$\frac{d\mu}{\mu}\sqrt{\frac{\mu^2-a^2}{\mu^2-c^2}}\mp\frac{d\nu}{\nu}\sqrt{\frac{a^2-\nu^2}{c^2-\nu^2}}=0.$$

It is to be observed that we cannot pass from the form (23) to the form of Uf in Cartesian coordinates by equations (5) solved for μ and ν in terms of x and y, since the equations (5) do not represent a continuous group of transformations.

PRINCETON, NEW JERSEY, *October* 12, 1897.

Note on the Integration of a certain System of Differential Equations.

By John Eiesland.

Professor Craig, in an article which appears in the present volume of this Journal and is entitled "Displacements depending on One, Two and Three Variables in a Space of Four Dimensions," deduces the following system of differential equations:

$$\left.\begin{aligned}
\frac{d\alpha}{dt} &= p_{12}\beta - p_{13}\gamma + p_{14}\delta, \\
\frac{d\beta}{dt} &= -p_{12}\alpha + p_{23}\gamma + p_{24}\delta, \\
\frac{d\gamma}{dt} &= p_{13}\alpha - p_{23}\beta + p_{34}\delta, \\
\frac{d\delta}{dt} &= -p_{14}\alpha - p_{24}\beta - p_{34}\gamma,
\end{aligned}\right\} \tag{1}$$

which are satisfied by the four groups of direction cosines a, b, c, d; a', b', c', d'; a'', b'', c'', d''; a''', b''', c''', d''' and admits of the integral

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \text{const.}$$

We shall first consider the case where the constant differs from zero. By dividing the left side by a suitable constant, we may always suppose that

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1. \tag{2}$$

We now employ the transformation

$$\alpha = \frac{\lambda}{\sqrt{x^2+1}}, \quad \beta = \frac{\mu}{\sqrt{x^2+1}}, \quad \gamma = \frac{\nu}{\sqrt{x^2+1}}, \quad \delta = \frac{1}{\sqrt{x^2+1}},$$

where $x^2 = \lambda^2 + \mu^2 + \nu^2$. The system (1) now takes the form

$$\left.\begin{aligned}
\frac{d\lambda}{dt} &= p_{14} + p_{12}\mu - p_{13}\nu + \lambda(p_{14}\lambda + p_{24}\mu + p_{34}\nu), \\
\frac{d\mu}{dt} &= p_{24} - p_{12}\lambda + p_{23}\nu + \mu(p_{14}\lambda + p_{24}\mu + p_{34}\nu), \\
\frac{d\nu}{dt} &= p_{34} + p_{13}\lambda - p_{23}\mu + \nu(p_{14}\lambda + p_{24}\mu + p_{34}\nu),
\end{aligned}\right\} \tag{3}$$

which is a generalization of Ricatti's equation.

If now we consider λ, μ, ν as coordinates in ordinary space and t as time, this system defines an infinitesimal transformation:

$$\begin{aligned}
Uf = &\ \{p_{14} + p_{12}\mu - p_{13}\nu + \lambda(p_{14}\lambda + p_{24}\mu + p_{34}\nu)\}\frac{\partial f}{\partial\lambda} \\
&+ \{p_{24} - p_{12}\lambda + p_{23}\nu + \mu(p_{14}\lambda + p_{24}\mu + p_{34}\nu)\}\frac{\partial f}{\partial\mu} \\
&+ \{p_{34} + p_{13}\lambda - p_{23}\mu + \nu(p_{14}\lambda + p_{24}\mu + p_{34}\nu)\}\frac{\partial f}{\partial\nu},
\end{aligned} \tag{4}$$

which is performed on the point λ, μ, ν in the element of time dt. A point λ, μ, ν is thus transformed from an initial position λ_0, μ_0, ν_0 into another general position λ, μ, ν by means of a projective transformation which changes with the time t. The general integral of (3) will therefore have the form of a finite projective transformation

$$\begin{gathered}
\lambda = \frac{a_1\lambda_0 + b_1\mu_0 + c_1\nu_0 + d_1}{a_4\lambda_0 + b_4\mu_0 + c_4\nu_0 + d_4}, \quad \mu = \frac{a_2\lambda_0 + b_2\mu_0 + c_2\nu_0 + d_2}{a_4\lambda_0 + b_4\mu_0 + c_4\nu_0 + d_4}, \\
\nu = \frac{a_3\lambda_0 + b_3\mu_0 + c_3\nu_0 + d_3}{a_4\lambda_0 + b_4\mu_0 + c_4\nu_0 + d_4},
\end{gathered} \tag{5}$$

where the coefficients a_i, b_i, c_i, d_i, $(i = 1, 2, 3, 4)$ are functions of t, and λ_0, μ_0, ν_0 are the coordinates of the initial point and play the role of arbitrary constants of integration. It is easily seen that the coefficients a_i, b_i, c_i, d_i are particular solutions of the system (1). In fact, the general integrals of this are

$$\begin{aligned}
\alpha &= k_1a_1 + k_2b_1 + k_3c_1 + k_4d_1, \\
\beta &= k_1a_2 + k_2b_2 + k_3c_2 + k_4d_2, \\
\gamma &= k_1a_3 + k_2b_3 + k_3c_3 + k_4d_3, \\
\delta &= k_1a_4 + k_2b_4 + k_3c_4 + k_4d_4,
\end{aligned}$$

where $a_1, b_1, c_1, d_1; a_2, b_2, c_2, d_2; a_3, b_3, c_3, d_3; a_4, b_4, c_4, d_4$ form a set of four particular solutions of (1), and from this system of integrals we readily pass to (5) by taking into account the relations

$$\lambda = \frac{\alpha}{\delta}, \quad \mu = \frac{\beta}{\delta}, \quad \nu = \frac{\gamma}{\delta}.$$

Suppose now that we know a set of three particular solutions of the system (3). Let this be $\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3; \nu_1, \nu_2, \nu_3$, the determinant of which we suppose different from zero. We are able to express the general system of integrals by means of these solutions.

An easy calculation will show that this system is

$$\left.\begin{aligned}
\lambda &= \frac{\dfrac{\lambda_1}{\sqrt{x_1^2+1}}\lambda_0 + \dfrac{\lambda_2}{\sqrt{x_2^2+1}}\mu_0 + \dfrac{\lambda_3}{\sqrt{x_3^2+1}}\nu_0 + \dfrac{\lambda_4}{\sqrt{x_4^2+1}}}{\dfrac{1}{\sqrt{x_1^2+1}}\lambda_0 + \dfrac{1}{\sqrt{x_2^2+1}}\mu_0 + \dfrac{1}{\sqrt{x_3^2+1}}\nu_0 + \dfrac{1}{\sqrt{x_4^2+1}}}, \\
\mu &= \frac{\dfrac{\mu_1}{\sqrt{x_1^2+1}}\lambda_0 + \dfrac{\mu_2}{\sqrt{x_2^2+1}}\mu_0 + \dfrac{\mu_3}{\sqrt{x_3^2+1}}\nu_0 + \dfrac{\mu_4}{\sqrt{x_4^2+1}}}{\dfrac{1}{\sqrt{x_1^2+1}}\lambda_0 + \dfrac{1}{\sqrt{x_2^2+1}}\mu_0 + \dfrac{1}{\sqrt{x_3^2+1}}\nu_0 + \dfrac{1}{\sqrt{x_4^2+1}}}, \\
\nu &= \frac{\dfrac{\nu_1}{\sqrt{x_1^2+1}}\lambda_0 + \dfrac{\nu_2}{\sqrt{x_2^2+1}}\mu_0 + \dfrac{\nu_3}{\sqrt{x_3^2+1}}\nu_0 + \dfrac{\nu_4}{\sqrt{x_4^2+1}}}{\dfrac{1}{\sqrt{x_1^2+1}}\lambda_0 + \dfrac{1}{\sqrt{x_2^2+1}}\mu_0 + \dfrac{1}{\sqrt{x_3^2+1}}\nu_0 + \dfrac{1}{\sqrt{x_4^2+1}}},
\end{aligned}\right\} \tag{6}$$

where

$$\lambda_4 = \frac{\begin{vmatrix} \mu_1 & \nu_1 & 1 \\ \mu_2 & \nu_2 & 1 \\ \mu_3 & \nu_3 & 1 \end{vmatrix}}{\Delta}, \quad \mu_4 = \frac{\begin{vmatrix} \lambda_1 & \nu_1 & 1 \\ \lambda_2 & \nu_2 & 1 \\ \lambda_3 & \nu_3 & 1 \end{vmatrix}}{\Delta}, \quad \nu_4 = \frac{\begin{vmatrix} \lambda_1 & \mu_1 & 1 \\ \lambda_2 & \mu_2 & 1 \\ \lambda_3 & \mu_3 & 1 \end{vmatrix}}{\Delta}, \quad \Delta = \begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}.$$

Returning now to the system (3), we will consider λ, μ, ν, t as coordinates in a four-dimensional space. An integral hypersurface may be represented by the equation

$$\nu = \rho\lambda + \sigma\mu + \tau, \tag{7}$$

ρ, σ, and τ being certain unknown functions of t. Differentiating and substituting the values of $\frac{d\lambda}{dt}, \frac{d\mu}{dt}, \frac{d\nu}{dt}$ given by (3), we get

$$
\begin{aligned}
& p_{34} + p_{13}\lambda - p_{23}\mu + \nu\left(p_{14}\lambda + p_{24}\mu + p_{34}\nu\right) \\
& - \rho\left\{p_{14} + p_{12}\mu - p_{13}\mu + \lambda\left(p_{14}\lambda + p_{24}\mu + p_{34}\nu\right)\right\} \\
& - \sigma\left\{p_{24} - p_{12}\lambda + p_{23}\nu + \mu\left(p_{14}\lambda + p_{24}\mu + p_{34}\nu\right)\right\} \\
& - \lambda\frac{d\rho}{dt} - \mu\frac{d\sigma}{dt} - \frac{d\tau}{dt} = 0,
\end{aligned}
$$

and, since $\nu = \rho\lambda + \sigma\mu + \tau$ is an integral hypersurface, this equation must be identically zero for all values of λ and μ, when we substitute for ν its value given by (7). We thus arrive at the following system of differential equations satisfied by the functions ρ, σ and τ:

$$
\left.
\begin{aligned}
\frac{d\rho}{dt} &= p_{13} + p_{12}\sigma + p_{14}\tau + \rho\left(p_{13}\rho - p_{23}\sigma + p_{34}\tau\right), \\
\frac{d\sigma}{dt} &= - p_{23} - p_{12}\rho + p_{24}\sigma + \sigma\left(p_{13}\rho - p_{23}\sigma + p_{34}\tau\right), \\
\frac{d\tau}{dt} &= p_{34} - p_{14}\rho - p_{24}\sigma + \tau\left(p_{13}\rho - p_{23}\sigma + p_{34}\tau\right),
\end{aligned}
\right\}
\tag{8}
$$

a system analogous to (3).

Suppose now that we know a particular solution of (8); then the integral hypersurface is known, and we shall prove that *the system* (3) *can be reduced to a linear system in three variables.* Let the surface be written as before

$$\nu = \rho\lambda + \sigma\mu + \tau,$$

where ρ, σ, τ form a particular solution of (8). If we intersect this surface by the linear spaces $t = \text{const.}$, we get an infinity of planes which stand in a projective relation to each other by virtue of the transformation (4). We now introduce a projective transformation of coordinates, whereby the plane of each space $t = \text{const.}$ is transformed to infinity. Such a transformation is

$$\lambda_1 = \frac{\lambda}{\nu - \rho\lambda - \sigma\mu - \tau}, \quad \mu_1 = \frac{\mu}{\nu - \rho\lambda - \sigma\mu - \tau}, \quad \nu_1 = \frac{1}{\nu - \rho\lambda - \sigma\mu - \tau}. \tag{9}$$

Introducing these new variables in (3), taking also account of the equations (8), we get the following linear system:

$$
\left.
\begin{aligned}
\frac{d\lambda_1}{dt} &= - p_{13} - \left(p_{34}\tau + 2p_{13}\rho - p_{23}\sigma\right)\lambda_1 + \left(p_{12} - p_{13}\sigma\right)\mu_1 + \left(p_{14} - p_{13}\tau\right)\nu_1, \\
\frac{d\mu_1}{dt} &= p_{23} - \left(p_{12} - p_{23}\rho\right)\lambda_1 - \left(p_{24}\tau - 2p_{23}\sigma + p_{23}\rho\right)\mu_1 + \left(p_{24} + p_{23}\tau\right)\nu_1, \\
\frac{d\nu_1}{dt} &= - p_{34} - \left(p_{14} + p_{34}\rho\right)\lambda_1 - \left(p_{24} + p_{34}\sigma\right)\mu_1 - \left(p_{13}\rho + 2p_{34}\tau - p_{23}\sigma\right)\nu.
\end{aligned}
\right\}
\tag{10}
$$

A knowledge of a particular solution of (8) has thus reduced the integration of (1) to the integration of the linear system (10), which we shall write in the general form

$$\left.\begin{aligned}\frac{d\lambda_1}{dt} &= a_1 + a_2\lambda_1 + a_3\mu_1 + a_4\nu_1,\\ \frac{d\mu_1}{dt} &= b_1 + b_2\lambda_1 + b_3\mu_1 + b_4\nu_1,\\ \frac{d\nu_1}{dt} &= c_1 + c_2\lambda_1 + c_3\mu_1 + c_4\nu_1.\end{aligned}\right\} \qquad (11)$$

We now proceed in the same way as before, writing

$$\nu_1 = \rho_1\lambda_1 + \sigma_1\mu_1 + \tau_1,$$

where ρ_1, σ_1, τ_1 are unknown functions of t. Differentiating and substituting the values of $\frac{d\lambda_1}{dt}$, $\frac{d\mu_1}{dt}$, $\frac{d\nu_1}{dt}$, we get

$$\begin{aligned}c_1 + c_2\lambda + c_3\mu_1 + c_4\nu_1 - \rho_1(a_1 + a_2\lambda_1 + a_3\mu_1 + a_4\nu_1)&\\ - \sigma_1(b_1 + b_2\lambda_1 + b_3\mu_1 + b_4\nu_1) - \lambda_1\frac{d\rho_1}{dt} - \mu_1\frac{d\sigma_1}{dt} - \frac{d\tau}{dt} &= 0,\end{aligned}$$

and this equation must be identically zero for all values of λ_1 and μ_1 when we substitute for ν_1 its value. This gives us the following system of differential equations:

$$\left.\begin{aligned}\frac{d\rho_1}{dt} &= c_2 - (a_2 - c_4)\rho_1 - b_2\sigma_1 - \rho_1(a_4\rho_1 + b_4\sigma_1),\\ \frac{d\sigma_1}{dt} &= c_3 - a_3\rho_1 - (b_3 - c_4)\sigma_1 - \sigma_1(a_4\rho_1 + b_4\sigma_1),\end{aligned}\right\} \qquad (12)$$

$$\frac{d\tau_1}{dt} = c_1 - a_1\rho_1 - b_1\sigma_1 - c_4\tau_1 - \tau_1(a_4\rho_1 + b_4\sigma_1).$$

Let us suppose that the first two equations, which do not contain the variable τ_1, have been integrated. Substituting the values of ρ_1 and σ_1 in the third it reduces to the form

$$\frac{d\tau_1}{dt} = A\tau_1 + B,$$

which may be integrated by two quadratures. The general integrals ρ_1, σ_1, τ_1 having been found, all the integral hypersurfaces $\nu_1 = \rho_1\lambda_1 + \sigma_1\mu_1 + \tau_1$ are known, and therefore also all the integral curves. We may therefore say: *If a particular solution of the system* (8) *is known, the integration of* (3) *is reduced to the integration of a system of generalized Ricatti's equations in two variables and two quadratures.*

We may now proceed with the system (12), using a method the one employed for three variables.* An integral surface may b

$$\sigma_1 = \phi\rho_1 + \psi,$$

where ϕ and ψ are particular solutions of the equations

$$\frac{d\phi}{dt} = -a_3 + (a_2 - b_3)\phi - a_4\psi - \phi(b_4\psi - b_2\phi),$$

$$\frac{d\psi}{dt} = c_3 - c_2\phi - (b_3 - c_4)\psi - \phi(b_4\psi - b_2\phi).$$

Employing the transformation

$$\rho_1' = \frac{\rho_1}{\sigma_1 - \phi\rho_1 - \psi}, \quad \sigma_1' = \frac{1}{\sigma_1 - \phi\rho_1 - \psi},$$

we finally arrive at the linear system

$$\frac{d\rho_1'}{dt} = A_1\rho_1' + B_1\sigma_1' + C_1,$$

$$\frac{d\sigma_1'}{dt} = A_2\rho_1' + B_2\sigma_1' + C_2,$$

the integration of which can by a method identical with the one em case of three variables be reduced to the integration of a Ricatti single variable and two quadratures. We may therefore say, *i solution ρ, σ, τ of the system* (8) *and also a particular solution ϕ, ψ* (14) *are known, the integration of the original system* (1) *is reduced to of a Ricatti equation and four quadratures.*

II. We shall now consider the case where $\alpha, \beta, \gamma, \delta$ is a set solutions satisfying the relation

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0,$$

and let us suppose that the p's are imaginary functions of t. In th able to find a transformation by means of which the problem of simplified a great deal.

We put

$$\left.\begin{aligned} \frac{\alpha + i\beta}{\gamma - i\delta} &= -\frac{(\gamma + i\delta)}{\alpha - i\beta} = x, \\ \frac{\alpha - i\beta}{\gamma - i\delta} &= -\frac{\gamma + i\delta}{\alpha + i\beta} = -\frac{1}{y}, \end{aligned}\right\}$$

---

* For a complete discussion of the generalized system of Ricatti's equations in Sophus Lie's "Vorlesungen über continuierliche Gruppen," p. 778. See also an article "Zur Theorie der Differentialgleichungen," in Crelle's Journal, vol. 115.

which give us the relations

$$-i\frac{1-xy}{x-y}=\frac{\alpha}{\delta},\quad \frac{1+xy}{x-y}=\frac{\beta}{\delta},\quad i\frac{x+y}{x-y}=\frac{\gamma}{\delta},$$

or, introducing a factor of proportionality ρ,

$$\alpha=-\rho i(1-xy),\quad \beta=\rho(1+xy),\quad \gamma=\rho i(x+y),\quad \delta=\rho(x-y).$$

Differentiating, we get

$$\begin{aligned}
\frac{d\alpha}{dt}&=-i(1-xy)\frac{d\rho}{dt}+\rho i\left(y\frac{dx}{dt}+x\frac{dy}{dt}\right),\\
\frac{d\beta}{dt}&=\quad(1+xy)\frac{d\rho}{dt}+\rho\left(y\frac{dx}{dt}+x\frac{dy}{dt}\right),\\
\frac{d\gamma}{dt}&=\quad i(x+y)\frac{d\rho}{dt}+\rho i\left(\frac{dx}{dt}+\frac{dy}{dt}\right),\\
\frac{d\delta}{dt}&=\quad(x-y)\frac{d\rho}{dt}+\rho\left(\frac{dx}{dt}-\frac{dy}{dt}\right).
\end{aligned}$$

The first two of these equations give us at once

$$-2i\frac{d\rho}{dt}=\frac{d\alpha}{dt}-i\frac{d\beta}{dt}.$$

Substituting this value of $\frac{d\rho}{dt}$ in the last two equations and solving for $\frac{dx}{dt}$ and $\frac{dy}{dt}$, we get

$$\begin{aligned}
2i\rho\frac{dx}{dt}&=\frac{d\gamma}{dt}+i\frac{d\delta}{dt}+x\left(\frac{d\alpha}{dt}-i\frac{d\beta}{dt}\right),\\
2i\rho\frac{dy}{dt}&=\frac{d\gamma}{dt}-i\frac{d\delta}{dt}+y\left(\frac{d\alpha}{dt}-i\frac{d\beta}{dt}\right).
\end{aligned}$$

Introducing on the right-hand side the variables x, y and ρ and reducing we get the following equations:

$$\left.\begin{aligned}
\frac{dx}{dt}&=-i(p_{12}-p_{34})x+\frac{(p_{14}+p_{23})i-(p_{13}+p_{24})}{2}\\
&\qquad+\frac{-(p_{13}+p_{24})-(p_{14}+p_{23})i}{2}x^2,\\
\frac{dy}{dt}&=-i(p_{12}-p_{34})y+\frac{(p_{23}-p_{14})i-(p_{13}-p_{24})}{2}\\
&\qquad+\frac{-(p_{13}-p_{24})-(p_{23}-p_{14})i}{2}y^2,
\end{aligned}\right\}\quad(16)$$

$$\frac{i}{\rho}\frac{d\rho}{dt}=-p_{12}+\frac{(p_{24}+p_{13})i-(p_{14}+p_{23})}{2}x+\frac{p_{14}-p_{23}+(p_{13}-p_{24})i}{2}y.$$

The equations in x and y have the well-known Ricatti's form, and resemble those deduced by Darboux in his "Leçons sur la Théorie des Surfaces," vol. I, p. 22. We now put

$$u = x + \frac{1}{\lambda}, \quad v = y + \frac{1}{\mu},$$

the above-mentioned equations will take the form

$$\frac{d\lambda}{dt} = \frac{-(p_{13}+p_{24})-(p_{14}+p_{23})\,i}{2} + [i\,(p_{12}-p_{34}) + \{(p_{13}+p_{24})+(p_{14}+p_{23})\,i\}\,x]\,\lambda,$$

$$\frac{d\mu}{dt} = \frac{-(p_{13}-p_{24})-(p_{23}-p_{14})\,i}{2} + [i\,(p_{12}-p_{34}) + \{(p_{13}-p_{24})+(p_{23}-p_{14})\,i\}\,y]\,\mu,$$

which may be solved by means of 4 quadratures. We have thus arrived at the following result: *If α, β, γ, δ be a particular solution of* (1) *satisfying the relation* $\alpha^2+\beta^2+\gamma^2+\delta^2=0$, *and if also the p's are imaginary functions of t, then four quadratures will be necessary for the complete integration of* (1).

Suppose that the p's are real; if x be a particular solution of the equation in x, then will $-\frac{1}{x'}$, where x' denotes the conjugate imaginary of x, also be a particular solution of the same equation. This is easily seen by changing i into $-i$ on the right-hand side and putting x' for x. (Compare Darboux, "Leçons," p. 23.) The same holds for the equation in y, so that we get two pairs of particular solutions $x, -\frac{1}{x'}$; $y, -\frac{1}{y'}$. But we know that if two solutions of a Ricatti equation are known, only one quadrature is necessary for the complete integration of the equation. It follows therefore that when the p's are real functions of t, only two quadratures are necessary for the complete integration of (1), α, β, γ, δ being a given particular solution.

Compound Determinants.

By William H. Metzler, Ph. D.

It is proposed in this paper to show, by a method similar to that employed in the Am. Jour. of Math., vol. XVI, No. 3, pp. 131–150, how the value of certain minors* of the m^{th} compound of a given determinant may very easily be found in terms of the given determinant and its minors.

1. If Δ denote a determinant of order n, then $\Delta\,(m)$ will denote the m^{th} compound of Δ, and $\Delta\,(n-m)$ will be termed the adjugate of $\Delta\,(m)$. We have the well-known relation connecting minors of $\Delta\,(m)$ with those of $\Delta\,(n-m)$, viz. *any minor of $\Delta\,(m)$ of order k is equal to the complementary of the corresponding minor of $\Delta\,(n-m)$ multiplied by $\Delta^{k-(n-1)_m}$.*†

2. For the sake of definiteness, let us start with a determinant of order five:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix},$$

then

$$\Delta_{(3)} = \begin{vmatrix}
A_{\substack{123\\123}} & A_{\substack{123\\124}} & A_{\substack{123\\125}} & A_{\substack{123\\134}} & A_{\substack{123\\135}} & A_{\substack{123\\145}} & A_{\substack{123\\234}} & A_{\substack{123\\235}} & A_{\substack{123\\245}} & A_{\substack{123\\345}} \\
A_{\substack{124\\123}} & A_{\substack{124\\124}} & A_{\substack{124\\125}} & A_{\substack{124\\134}} & A_{\substack{124\\135}} & A_{\substack{124\\145}} & A_{\substack{124\\234}} & A_{\substack{124\\235}} & A_{\substack{124\\245}} & A_{\substack{124\\345}} \\
A_{\substack{125\\123}} & A_{\substack{125\\124}} & A_{\substack{125\\125}} & A_{\substack{125\\134}} & A_{\substack{125\\135}} & A_{\substack{125\\145}} & A_{\substack{125\\234}} & A_{\substack{125\\235}} & A_{\substack{125\\245}} & A_{\substack{125\\345}} \\
A_{\substack{134\\123}} & A_{\substack{134\\124}} & A_{\substack{134\\125}} & A_{\substack{134\\134}} & A_{\substack{134\\135}} & A_{\substack{134\\145}} & A_{\substack{134\\234}} & A_{\substack{134\\235}} & A_{\substack{134\\245}} & A_{\substack{134\\345}} \\
A_{\substack{135\\123}} & A_{\substack{135\\124}} & A_{\substack{135\\125}} & A_{\substack{135\\134}} & A_{\substack{135\\135}} & A_{\substack{135\\145}} & A_{\substack{135\\234}} & A_{\substack{135\\235}} & A_{\substack{135\\245}} & A_{\substack{135\\345}} \\
A_{\substack{145\\123}} & A_{\substack{145\\124}} & A_{\substack{145\\125}} & A_{\substack{145\\134}} & A_{\substack{145\\135}} & A_{\substack{145\\145}} & A_{\substack{145\\234}} & A_{\substack{145\\235}} & A_{\substack{145\\245}} & A_{\substack{145\\345}} \\
A_{\substack{234\\123}} & A_{\substack{234\\124}} & A_{\substack{234\\125}} & A_{\substack{234\\134}} & A_{\substack{234\\135}} & A_{\substack{234\\145}} & A_{\substack{234\\234}} & A_{\substack{234\\235}} & A_{\substack{234\\245}} & A_{\substack{234\\345}} \\
A_{\substack{235\\123}} & A_{\substack{235\\124}} & A_{\substack{235\\125}} & A_{\substack{235\\134}} & A_{\substack{235\\135}} & A_{\substack{235\\145}} & A_{\substack{235\\234}} & A_{\substack{235\\235}} & A_{\substack{235\\245}} & A_{\substack{235\\345}} \\
A_{\substack{245\\123}} & A_{\substack{245\\124}} & A_{\substack{245\\125}} & A_{\substack{245\\134}} & A_{\substack{245\\135}} & A_{\substack{245\\145}} & A_{\substack{245\\234}} & A_{\substack{245\\235}} & A_{\substack{245\\245}} & A_{\substack{245\\345}} \\
A_{\substack{345\\123}} & A_{\substack{345\\124}} & A_{\substack{345\\125}} & A_{\substack{345\\134}} & A_{\substack{345\\135}} & A_{\substack{345\\145}} & A_{\substack{345\\234}} & A_{\substack{345\\235}} & A_{\substack{345\\245}} & A_{\substack{345\\345}}
\end{vmatrix},$$

* All those which are expressible as a product of minors of the given determinant.

† Vide Muir's "Theory of Determinants," §175.

$$\Delta_{(3)} = \begin{vmatrix} A_{\substack{12\\12}} & A_{\substack{12\\13}} & A_{\substack{12\\14}} & A_{\substack{12\\15}} & A_{\substack{12\\23}} & A_{\substack{12\\24}} & A_{\substack{12\\25}} & A_{\substack{12\\34}} & A_{\substack{12\\35}} & A_{\substack{12\\45}} \\ A_{\substack{13\\12}} & A_{\substack{13\\13}} & A_{\substack{13\\14}} & A_{\substack{13\\15}} & A_{\substack{13\\23}} & A_{\substack{13\\24}} & A_{\substack{13\\25}} & A_{\substack{13\\34}} & A_{\substack{13\\35}} & A_{\substack{13\\45}} \\ A_{\substack{14\\12}} & A_{\substack{14\\13}} & A_{\substack{14\\14}} & A_{\substack{14\\15}} & A_{\substack{14\\23}} & A_{\substack{14\\24}} & A_{\substack{14\\25}} & A_{\substack{14\\34}} & A_{\substack{14\\35}} & A_{\substack{14\\45}} \\ A_{\substack{15\\12}} & A_{\substack{15\\13}} & A_{\substack{15\\14}} & A_{\substack{15\\15}} & A_{\substack{15\\23}} & A_{\substack{15\\24}} & A_{\substack{15\\25}} & A_{\substack{15\\34}} & A_{\substack{15\\35}} & A_{\substack{15\\45}} \\ A_{\substack{23\\12}} & A_{\substack{23\\13}} & A_{\substack{23\\14}} & A_{\substack{23\\15}} & A_{\substack{23\\23}} & A_{\substack{23\\24}} & A_{\substack{23\\25}} & A_{\substack{23\\34}} & A_{\substack{23\\35}} & A_{\substack{23\\45}} \\ A_{\substack{24\\12}} & A_{\substack{24\\13}} & A_{\substack{24\\14}} & A_{\substack{24\\15}} & A_{\substack{24\\23}} & A_{\substack{24\\24}} & A_{\substack{24\\25}} & A_{\substack{24\\34}} & A_{\substack{24\\35}} & A_{\substack{24\\45}} \\ A_{\substack{25\\12}} & A_{\substack{25\\13}} & A_{\substack{25\\14}} & A_{\substack{25\\15}} & A_{\substack{25\\23}} & A_{\substack{25\\24}} & A_{\substack{25\\25}} & A_{\substack{25\\34}} & A_{\substack{25\\35}} & A_{\substack{25\\45}} \\ A_{\substack{34\\12}} & A_{\substack{34\\13}} & A_{\substack{34\\14}} & A_{\substack{34\\15}} & A_{\substack{34\\23}} & A_{\substack{34\\24}} & A_{\substack{34\\25}} & A_{\substack{34\\34}} & A_{\substack{34\\35}} & A_{\substack{34\\45}} \\ A_{\substack{35\\12}} & A_{\substack{35\\13}} & A_{\substack{35\\14}} & A_{\substack{35\\15}} & A_{\substack{35\\23}} & A_{\substack{35\\24}} & A_{\substack{35\\25}} & A_{\substack{35\\34}} & A_{\substack{35\\35}} & A_{\substack{35\\45}} \\ A_{\substack{45\\12}} & A_{\substack{45\\13}} & A_{\substack{45\\14}} & A_{\substack{45\\15}} & A_{\substack{45\\23}} & A_{\substack{45\\24}} & A_{\substack{45\\25}} & A_{\substack{45\\34}} & A_{\substack{45\\35}} & A_{\substack{45\\45}} \end{vmatrix}$$

and

$$\Delta_{(4)} = \begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{vmatrix}.$$

By Laplace's theorem we have

$$\left.\begin{aligned} A_{55} &= a_{11}A_{\substack{15\\15}} - a_{12}A_{\substack{15\\25}} + a_{13}A_{\substack{15\\35}} - a_{14}A_{\substack{15\\45}}, \\ &= -a_{21}A_{\substack{25\\15}} + a_{22}A_{\substack{25\\25}} - a_{23}A_{\substack{25\\35}} + a_{24}A_{\substack{25\\45}} = \text{etc.}, \\ &= a_{11}A_{\substack{15\\15}} - a_{21}A_{\substack{25\\15}} + a_{31}A_{\substack{35\\15}} - a_{41}A_{\substack{45\\15}} = \text{etc.}, \\ A_{24} &= a_{11}A_{\substack{12\\14}} - a_{12}A_{\substack{12\\24}} + a_{13}A_{\substack{12\\34}} - a_{15}A_{\substack{12\\45}} = \text{etc.} \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} 0 &= -a_{31}A_{\substack{15\\15}} + a_{32}A_{\substack{15\\25}} - a_{33}A_{\substack{15\\35}} + a_{34}A_{\substack{15\\45}}, \\ &= a_{31}A_{\substack{15\\15}} - a_{32}A_{\substack{15\\25}} + a_{33}A_{\substack{15\\35}} - a_{34}A_{\substack{15\\45}} = \text{etc.}, \\ &= -a_{12}A_{\substack{15\\15}} + a_{22}A_{\substack{25\\15}} - a_{32}A_{\substack{35\\15}} + a_{42}A_{\substack{45\\15}} = \text{etc.} \end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned} A_{\substack{45\\45}} &= a_{33}A_{\substack{345\\345}} - a_{32}A_{\substack{345\\245}} + a_{31}A_{\substack{345\\145}} = \text{etc.}, \\ &= a_{33}A_{\substack{345\\345}} - a_{23}A_{\substack{245\\345}} + a_{13}A_{\substack{145\\345}} = \text{etc.}, \\ A_{\substack{24\\35}} &= a_{11}A_{\substack{124\\135}} - a_{12}A_{\substack{124\\235}} + a_{14}A_{\substack{124\\345}} = \text{etc.} \end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned} 0 &= -a_{23}A_{\substack{345\\345}} + a_{22}A_{\substack{345\\245}} - a_{21}A_{\substack{345\\145}} = \text{etc.},\\ &= a_{51}A_{\substack{124\\135}} - a_{52}A_{\substack{124\\235}} + a_{51}A_{\substack{124\\345}} = \text{etc.} \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} A_{55} &= A_{\substack{345\\345}}A_{\substack{125\\125}} - A_{\substack{345\\245}}A_{\substack{125\\135}} + A_{\substack{345\\235}}A_{\substack{125\\145}}\\ &\quad + A_{\substack{345\\145}}A_{\substack{125\\235}} - A_{\substack{345\\135}}A_{\substack{125\\245}} + A_{\substack{345\\125}}A_{\substack{125\\345}} = \text{etc.},\\ A_{33} &= A_{\substack{123\\123}}A_{\substack{345\\245}} - A_{\substack{123\\124}}A_{\substack{345\\235}} + A_{\substack{123\\125}}A_{\substack{345\\234}}\\ &\quad + A_{\substack{123\\234}}A_{\substack{345\\125}} - A_{\substack{123\\235}}A_{\substack{345\\125}} + A_{\substack{123\\245}}A_{\substack{345\\123}} = \text{etc.} \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} 0 &= A_{\substack{345\\345}}A_{\substack{135\\125}} - A_{\substack{345\\245}}A_{\substack{135\\135}} + A_{\substack{345\\235}}A_{\substack{135\\145}}\\ &\quad + A_{\substack{345\\145}}A_{\substack{135\\235}} - A_{\substack{345\\135}}A_{\substack{135\\245}} + A_{\substack{345\\125}}A_{\substack{135\\345}} = \text{etc.} \end{aligned}\right\} \tag{6}$$

Operating on these equations by the *Law of Complementaries** we get:

$$\left.\begin{aligned} \Delta a_{55} &= A_{11}A_{\substack{234\\234}} - A_{12}A_{\substack{234\\134}} + A_{13}A_{\substack{234\\124}} - A_{14}A_{\substack{234\\123}},\\ &= -A_{21}A_{\substack{134\\234}} + A_{22}A_{\substack{134\\134}} - A_{23}A_{\substack{134\\124}} + A_{24}A_{\substack{134\\123}} = \text{etc.},\\ &= A_{11}A_{\substack{234\\234}} - A_{21}A_{\substack{134\\234}} + A_{31}A_{\substack{124\\234}} - A_{41}A_{\substack{123\\234}} = \text{etc.},\\ \Delta a_{24} &= A_{11}A_{\substack{345\\235}} - A_{12}A_{\substack{345\\135}} + A_{13}A_{\substack{345\\125}} - A_{15}A_{\substack{345\\123}} = \text{etc.} \end{aligned}\right\} \tag{1'}$$

$$\left.\begin{aligned} 0 &= -A_{21}A_{\substack{234\\234}} + A_{22}A_{\substack{234\\134}} - A_{23}A_{\substack{234\\124}} + A_{24}A_{\substack{234\\123}} = \text{etc.},\\ &= -A_{12}A_{\substack{234\\234}} + A_{22}A_{\substack{134\\234}} - A_{32}A_{\substack{124\\234}} + A_{42}A_{\substack{123\\234}} = \text{etc.} \end{aligned}\right\} \tag{2'}$$

$$\left.\begin{aligned} \Delta A_{\substack{123\\123}} &= A_{33}A_{\substack{12\\12}} - A_{32}A_{\substack{12\\13}} + A_{31}A_{\substack{12\\23}} = \text{etc.}\\ &= A_{33}A_{\substack{12\\12}} - A_{23}A_{\substack{13\\12}} + A_{13}A_{\substack{23\\12}} = \text{etc.};\\ \Delta A_{\substack{135\\124}} &= A_{11}A_{\substack{35\\24}} - A_{12}A_{\substack{35\\14}} + A_{14}A_{\substack{35\\12}} = \text{etc.} \end{aligned}\right\} \tag{3'}$$

$$\left.\begin{aligned} 0 &= -A_{23}A_{\substack{12\\12}} + A_{22}A_{\substack{12\\13}} - A_{21}A_{\substack{12\\23}} = \text{etc.}\\ &= A_{51}A_{\substack{35\\24}} - A_{52}A_{\substack{35\\14}} + A_{54}A_{\substack{35\\12}} = \text{etc.} \end{aligned}\right\} \tag{4'}$$

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* Vide Muir, "Theory of Determinants," §98.

$$\left.\begin{aligned}\Delta A_{55} &= A_{\substack{12\\12}}A_{\substack{34\\34}} - A_{\substack{12\\13}}A_{\substack{34\\24}} + A_{\substack{12\\14}}A_{\substack{34\\23}} + A_{\substack{12\\23}}A_{\substack{34\\14}}\\ &\qquad - A_{\substack{12\\24}}A_{\substack{34\\13}} + A_{\substack{12\\34}}A_{\substack{34\\12}} = \text{etc.},\\ \Delta a_{33} &= A_{\substack{45\\45}}A_{\substack{12\\12}} - A_{\substack{45\\35}}A_{\substack{12\\14}} + A_{\substack{45\\34}}A_{\substack{12\\15}} + A_{\substack{45\\15}}A_{\substack{12\\34}}\\ &\qquad - A_{\substack{45\\14}}A_{\substack{12\\35}} + A_{\substack{45\\13}}A_{\substack{12\\45}} = \text{etc.}\end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned}0 &= A_{\substack{12\\12}}A_{\substack{24\\34}} - A_{\substack{12\\13}}A_{\substack{24\\24}} + A_{\substack{12\\14}}A_{\substack{24\\23}} + A_{\substack{12\\23}}A_{\substack{24\\14}}\\ &\qquad - A_{\substack{12\\24}}A_{\substack{24\\13}} + A_{\substack{12\\34}}A_{\substack{24\\12}} = \text{etc.}\end{aligned}\right\} \tag{6'}$$

Operating on equations (3) and (4) by the *Law of Extensible Minors*,* we get

$$\left.\begin{aligned}a_{55}A_{44} &= A_{\substack{34\\34}}A_{\substack{124\\124}} - A_{\substack{34\\24}}A_{\substack{124\\134}} + A_{\substack{34\\14}}A_{\substack{124\\234}} = \text{etc.},\\ &= A_{\substack{34\\34}}A_{\substack{124\\124}} - A_{\substack{24\\34}}A_{\substack{134\\124}} + A_{\substack{14\\34}}A_{\substack{234\\124}} = \text{etc.},\\ a_{45}A_{23} &= A_{\substack{12\\13}}A_{\substack{235\\234}} - A_{\substack{12\\23}}A_{\substack{235\\134}} + A_{\substack{12\\34}}A_{\substack{235\\123}} = \text{etc.}\end{aligned}\right\} \tag{3''}$$

$$\left.\begin{aligned}0 &= - A_{\substack{34\\34}}A_{\substack{134\\124}} + A_{\substack{34\\24}}A_{\substack{134\\134}} - A_{\substack{34\\14}}A_{\substack{134\\234}} = \text{etc.},\\ &= A_{\substack{13\\13}}A_{\substack{123\\234}} - A_{\substack{12\\23}}A_{\substack{123\\134}} + A_{\substack{12\\34}}A_{\substack{123\\123}} = \text{etc.}\end{aligned}\right\} \tag{4''}$$

3. If for the sake of uniformity we write A for Δ, $A_{\substack{r\\s}}$ for Ars, $A_{\substack{23\,\dots\,n\\23\,\dots\,n}}$ for a_{11}, etc., then in each factor of every term of the foregoing expressions there are two lines of suffixes. I shall refer to them as the upper and lower. We may make the equations homogeneous in the A's by multiplying any term when necessary by $A_{\substack{123\,\dots\,n\\123\,\dots\,n}}$ which is unity.

4. If we are given any combination of n numbers k at a time, the combination of the remaining $(n-k)$ numbers is said to be the complementary with respect to n of the given combination. Considering any n numbers $a_1, a_2, a_3 \dots . a_n$, let $a_{m+1}, a_{m+2} \dots . a_n$ denote the combination complementary with respect to n of the combination $a_1, a_2, \dots . a_m$.

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* Muir, "Theory of Determinants," §179.

5. The equations of art. 2 are all of the form

$$A_{\substack{a_1 a_2 \ldots\ldots a_m c_1 c_2 \ldots\ldots c_i \\ b_1 b_2 \ldots\ldots b_m d_1 d_2 \ldots\ldots d_i}} A_{\substack{c_1 c_2 \ldots\ldots c_i \\ d_1 d_2 \ldots\ldots d_i}} = \sum (-1)^{\nu} A_{\substack{\alpha_1 \alpha_2 \ldots\ldots \alpha_r c_1 c_2 \ldots\ldots c_i \\ \beta_1 \beta_2 \ldots\ldots \beta_r d_1 d_2 \ldots\ldots d_i}} A_{\substack{\gamma_1 \gamma_2 \ldots\ldots \gamma_s c_1 c_2 \ldots\ldots c_i \\ \delta_1 \delta_2 \ldots\ldots \delta_s d_1 d_2 \ldots\ldots d_i}},^{*} \tag{A}$$

where $r + s = m$, and $\alpha_1, \alpha_2, \ldots\ldots \alpha_r$ are some r of the numbers $a_1, a_2, \ldots\ldots a_m$ and $\gamma_1, \gamma_2, \ldots\ldots \gamma_s$ are the remaining $m - r = s$ numbers, that is, $\alpha_1, \alpha_2, \ldots\ldots \alpha_r$ and $\gamma_1, \gamma_2, \ldots\ldots \gamma_s$ are complementary combinations of the numbers $a_1, a_2, \ldots a_m$; similarly $\beta_1, \beta_2, \ldots\ldots \beta_r$ and $\delta_1, \delta_2, \ldots\ldots \delta_s$ are complementary combinations of the numbers $b_1, b_2, \ldots\ldots b_m$. The c's are those numbers that are found in the upper line of the suffix of both factors of every term, and the d's are those numbers that are found in the lower line of the suffix of both factors of every term. Either the α's and γ's remain the same for every term while the β's and δ's vary, or the β's and δ's remain the same for every term while the α's and γ's vary. The numbers in the line of the suffixes which vary from term to term are the combinations r at a time with their complementaries of the m numbers $a_1, a_2 \ldots\ldots a_m$.

The value of ν for any term is given by the equation

$$\nu = \sum_1^r (\alpha_\kappa + \beta_\kappa) - \sum_1^{n-m} (\lambda_\kappa + \mu_\kappa),$$

where

$$\begin{aligned}
\lambda_1 &= \text{the number of } \alpha\text{'s} > a_{m+1}, \\
\lambda_2 &= \quad \text{"} \qquad \text{"} \quad \alpha\text{'s} > a_{m+2}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\lambda_\kappa &= \quad \text{"} \qquad \text{"} \quad \alpha\text{'s} > a_{m+\kappa}, \\
\mu_1 &= \quad \text{"} \qquad \text{"} \quad \beta\text{'s} > b_{m+1}, \\
\mu_2 &= \quad \text{"} \qquad \text{"} \quad \beta\text{'s} > b_{m+2}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\mu_\kappa &= \quad \text{"} \qquad \text{"} \quad \beta\text{'s} > b_{m+\kappa}.
\end{aligned}$$

If the c's and d's are the same, we may in practice neglect the second term in the value of ν.

If $n - m = 0$, then $\lambda_\kappa = \mu_\kappa = 0$.

If $i = 0$, that is, if there are no repetitions, $A_{\substack{c_1 c_2 \ldots\ldots c_i \\ d_1 d_2 \ldots d_i}}$ becomes A and equation (A) takes the form

$$A \, . \, A_{\substack{a_1 a_2 \ldots\ldots a_m \\ b_1 b_2 \ldots\ldots b_m}} = \sum (-1)^{\nu} A_{\substack{\alpha_1 \alpha_2 \ldots \alpha_r \\ \beta_1 \beta_2 \ldots\ldots \beta_r}} A_{\substack{\gamma_1 \gamma_2 \ldots\ldots \gamma_s \\ \delta_1 \delta_2 \ldots\ldots \delta_s}}, \tag{B}$$

which is the form of equations (1′), (3′), (5′).

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* The a's, the b's, the c's, etc., are supposed arranged in their natural order (order of magnitude).

6. Since the numbers in the upper line denote the rows and the numbers in the lower line denote the columns, it is evident that putting an α equal to a γ is equivalent to making two rows identical, and putting a β equal to a δ is equivalent to making two columns identical. If an α be put equal to a γ or a β to a δ, then two a's or two b's become identical and the expression vanishes, and the equation is of the form of (2), (4), (6), (2′), (4′), (6′), (4″).

7. Operating by the *Law of Complementaries* is equivalent to replacing each line of every suffix by its complementary with respect to n, and operating by the *Law of Extensible Minors* is equivalent to striking out the same j c's and the same j d's from the suffix of each factor of every term. It is possible therefore to reduce all equations to the form of equation (B). It may often happen that the same result is obtained by the operation of these two laws.

8. Let $(n|\underset{1}{m})$, $(n|\underset{2}{m})$, ... $(n|\underset{\mu}{m})$ represent the $n_m = \mu$ combinations of the n numbers 1, 2, 3, ... n taken m at a time,* and let $(\bar{n}|\underset{1}{m})$, $(\bar{n}|\underset{2}{m})$, ... $(\bar{n}|\underset{\mu}{m})$ denote the complementary combinations. If we take any combination of the numbers m at a time and combine the numbers in it in all possible ways l at a time, there would be $m_l = \lambda$ such combinations. Let

$$(n|\underset{\alpha}{m}|\underset{1}{l}),\ (n|\underset{\alpha}{m}|\underset{2}{l}),\ \dots\ (n|\underset{\alpha}{m}|\underset{\lambda}{l})$$

denote the λ combinations of the numbers in the combination $(n|\underset{\alpha}{m})$ taken l at a time, and let $(n|\underset{\alpha}{\bar{m}}|\underset{\beta}{l})$ denote the combination complementary with respect to m of the combination $(n|\underset{\alpha}{m}|\underset{\beta}{l})$, that is, the combination formed by the numbers remaining after the numbers in the combination $(n|\underset{\alpha}{m}|\underset{\beta}{l})$ are taken from the combination $(n|\underset{\alpha}{m})$. Let $(n|\underset{\alpha}{m}|\underset{\beta}{l})\ (\bar{n}|\underset{\alpha}{m})$ denote the combination of the numbers in the two combinations $(n|\underset{\alpha}{m}|\underset{\beta}{l})$ and $(\bar{n}|\underset{\alpha}{m})$, and so in general one combination following another will denote the combination of the numbers in the two combinations.

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* Let it always be understood that the numbers in every combination are arranged in their natural order.

9. With this notation, equation (B) may be written as follows:

$$\sum_{\beta \text{ or } \delta=1}^{\beta \text{ or } \delta=\lambda} (-1)^{\nu} A_{\substack{(n\,|\,m\,|\,l) \\ \alpha\;\;\beta \\ (n\,|\,m\,|\,l) \\ \gamma\;\;\delta}} A_{\substack{(n\,|\,\bar{m}\,|\,l) \\ \alpha\;\;\beta \\ (n\,|\,\bar{m}\,|\,l) \\ \gamma\;\;\delta}} = A \,.\, A_{\substack{(n\,|\,m) \\ \alpha \\ (n\,|\,m) \\ \gamma}}\,, \tag{C}$$

where $\sum\limits_{\beta \text{ or } \delta=1}^{\beta \text{ or } \delta=\lambda}$ means that when β varies from 1 to λ, δ is constant, and when δ varies from 1 to λ, β is constant. The value of ν is the same as before.

10. If we operate on equation (C) by the *Law of Complementaries* we get

$$\begin{aligned}
\sum_{\beta \text{ or } \delta=1}^{\beta \text{ or } \delta=\lambda} (-1)^{\nu} A_{\substack{(n\,|\,\bar{m}\,|\,l)(\bar{n}\,|\,m) \\ \alpha\;\;\beta\;\;\;\;\alpha \\ (n\,|\,\bar{m}\,|\,l)(\bar{n}\,|\,m) \\ \gamma\;\;\delta\;\;\;\;\gamma}} A_{\substack{(n\,|\,m\,|\,l)(\bar{n}\,|\,m) \\ \alpha\;\;\beta\;\;\;\;\alpha \\ (n\,|\,m\,|\,l)(\bar{n}\,|\,m) \\ \gamma\;\;\delta\;\;\;\;\gamma}} \\
= A_{\substack{(\bar{n}\,|\,m)(n\,|\,m) \\ \alpha\;\;\;\;\;\;\alpha \\ (\bar{n}\,|\,m)(n\,|\,m) \\ \gamma\;\;\;\;\;\;\gamma}} A_{\substack{(\bar{n}\,|\,m) \\ \alpha \\ (\bar{n}\,|\,m) \\ \gamma}} \\
= A_{\substack{(\bar{n}\,|\,m) \\ \alpha \\ (\bar{n}\,|\,m) \\ \gamma}}\,,
\end{aligned}$$

since $A_{\substack{(\bar{n}\,|\,m)(n\,|\,m) \\ \alpha\;\;\;\;\;\;\alpha \\ (n\,|\,m)(n\,|\,m) \\ \gamma\;\;\;\;\;\;\gamma}} = 1.$

This is the equation that would be obtained on expanding the minor $A_{\substack{(\bar{n}\,|\,m) \\ \alpha \\ (\bar{n}\,|\,m) \\ \gamma}}$ by Laplace's theorem.

11. Let A denote a determinant of order n, let $A_{(m)}$ denote the m^{th} compound of A, and let $A_{\substack{(n\,|\,m) \\ \alpha \\ (n\,|\,m) \\ \gamma}}^{(l)}$ denote the l^{th} compound of $A_{\substack{(n\,|\,m) \\ \alpha \\ (n\,|\,m) \\ \gamma}}$, etc.

The upper line of the suffix is the same for every constituent in the same row of $A_{(m)}$ and the lower line of the suffix is the same for every constituent in the same column.*

The upper lines of the suffixes of the constituents in the columns from top to bottom and the lower lines of the suffixes of the constituents in the rows from left to right are the combinations $(\bar{n}\,|\,\underset{1}{m})$, $(\bar{n}\,|\,\underset{2}{m}) \ldots . (\bar{n}\,|\,\underset{\mu}{m})$.

* If the rows and columns were interchanged, this statement would be reversed.

12. The determinant $A_{\substack{(\bar{n}|m)\\ \alpha\\ (\bar{n}|m)\\ \gamma}}^{(l)}$

$$\equiv \left(A_{\substack{(\bar{n}|m)(n|m|l)\\ \alpha \quad \alpha \; 1\\ (\bar{n}|m)(n|m|l)\\ \gamma \quad \gamma \; 1}} A_{\substack{(\bar{n}|m)(n|m|l)\\ \alpha \quad \alpha \; 2\\ (\bar{n}|m)(n|m|l)\\ \gamma \quad \gamma \; 2}} \cdots\cdots A_{\substack{(\bar{n}|m)(n|m|l)\\ \alpha \quad \alpha \; \lambda\\ (\bar{n}|m)(n|m|l)\\ \gamma \quad \gamma \; \lambda}} \right)$$

$$= A_{\substack{(\bar{n}|m)\\ \alpha\\ (\bar{n}|m)\\ \gamma}}^{(m-1)_{n-l-1}} ; = A_{\substack{(\bar{n}|m)\\ \alpha\\ (\bar{n}|m)\\ \gamma}}^{(m-1)_l} . \tag{7}$$

Operating on this equation by the *Law of Extensible Minors* we get

$$\left(A_{\substack{(n|m|l)\\ \alpha \; 1\\ (n|m|l)\\ \gamma \; 1}} A_{\substack{(n|m|l)\\ \alpha \; 2\\ (n|m|l)\\ \gamma \; 2}} \cdots\cdots A_{\substack{(n|m|l)\\ \alpha \; \lambda\\ (n|m|l)\\ \gamma \; \lambda}} \right)$$

$$= A^{(m-1)_l} . A_{\substack{(n|m)\\ \alpha\\ (n|m)\\ \gamma}}^{(m-1)_{l-1}} . \tag{8}$$

This is Sylvester's theorem,* since the constituents of the determinant on the left-hand side of the equation consist of the minor $A_{\substack{(n|m)\\ \alpha\\ (n|m)\\ \gamma}}$ bordered in all possible ways with l of the remaining rows and columns. It is evidently a minor of order λ of the determinant $A_{(n-l)}$, the $(n-l)^{\text{th}}$ compound of A.

Similarly $A_{\substack{(\bar{n}|m)\\ \alpha\\ (\bar{n}|m)\\ \gamma}}^{(m-l)}$

$$\equiv \left(A_{\substack{(\bar{n}|m)(n|\bar{m}|l)\\ \alpha \quad \alpha \; 1\\ (\bar{n}|m)(n|\bar{m}|l)\\ \gamma \quad \gamma \; 1}} A_{\substack{(\bar{n}|m)(n|\bar{m}|l)\\ \alpha \quad \alpha \; 2\\ (\bar{n}|m)(n|\bar{m}|l)\\ \gamma \quad \gamma \; 2}} \cdots\cdots A_{\substack{(\bar{n}|m)(n|\bar{m}|l)\\ \alpha \quad \alpha \; \lambda\\ (\bar{n}|m)(n|\bar{m}|l)\\ \gamma \quad \gamma \; \lambda}} \right)$$

$$= A_{\substack{(\bar{n}|m)\\ \alpha\\ (\bar{n}|m)\\ \gamma}}^{(m-1)_{l-1}} . \tag{9}$$

Operating on this by the same law we get

$$\left(A_{\substack{(n|\bar{m}|l)\\ \alpha \; 1\\ (n|\bar{m}|l)\\ \gamma \; 1}} A_{\substack{(n|\bar{m}|l)\\ \alpha \; 2\\ (n|\bar{m}|l)\\ \gamma \; 2}} \cdots\cdots A_{\substack{(n|\bar{m}|l)\\ \alpha \; \lambda\\ (n|\bar{m}|l)\\ \gamma \; \lambda}} \right)$$

$$= A^{(m-1)_{l-1}} A_{\substack{(n|m)\\ \alpha\\ (n|m)\\ \gamma}}^{(m-1)_l} . \tag{10}$$

Equation (4) might have been obtained from equation (1) by the *Law of Complementaries.*

---

* Vide Philosophical Magazine for 1851; also Camb. and Dublin Math. Jour., VIII, 60.

13. The k^{th} compound of $A_{\substack{(n\,|\,\underset{\alpha}{\bar m}\,|\,\underset{\beta}{l})\ (\bar n\,|\,\underset{\alpha}{m}) \\ (n\,|\,\underset{\gamma}{\bar m}\,|\,\underset{\delta}{l})\ (\bar n\,|\,\underset{\gamma}{m})}}$ is

$$\left(A_{\substack{(\bar n\,|\,\underset{\alpha}{m})(n\,|\,\underset{\alpha}{\bar m}\,|\,\underset{\beta}{l})(n\,|\,\underset{\alpha}{m}\,|\,\underset{\beta}{l}\,|\,\underset{1}{k}) \\ (\bar n\,|\,\underset{\gamma}{m})(n\,|\,\underset{\gamma}{\bar m}\,|\,\underset{\delta}{l})(n\,|\,\underset{\gamma}{m}\,|\,\underset{\delta}{l}\,|\,\underset{1}{k})}} A_{\substack{(\bar n\,|\,\underset{\alpha}{m})(n\,|\,\underset{\alpha}{\bar m}\,|\,\underset{\beta}{l})(\bar n\,|\,\underset{\alpha}{m}\,|\,\underset{\beta}{l}\,|\,\underset{2}{k}) \\ (\bar n\,|\,\underset{\gamma}{m})(n\,|\,\underset{\gamma}{\bar m}\,|\,\underset{\delta}{l})(n\,|\,\underset{\gamma}{m}\,|\,\underset{\delta}{l}\,|\,\underset{2}{k})}} \cdots\cdots A_{\substack{(\bar n\,|\,\underset{\alpha}{m})(n\,|\,\underset{\alpha}{\bar m}\,|\,\underset{\beta}{l})(n\,|\,\underset{\alpha}{m}\,|\,\underset{\beta}{l}\,|\,\underset{\sigma}{k}) \\ (\bar n\,|\,\underset{\gamma}{m})(n\,|\,\underset{\gamma}{\bar m}\,|\,\underset{\delta}{l})(n\,|\,\underset{\gamma}{m}\,|\,\underset{\delta}{l}\,|\,\underset{\sigma}{k})}}\right)_{\sigma=l_k}$$

and is equal to

$$A^{(l-1)_k}_{\substack{(\bar n\,|\,\underset{\alpha}{m})\,(n\,|\,\underset{\alpha}{\bar m}\,|\,\underset{\beta}{l}) \\ (\bar n\,|\,\underset{\gamma}{m})\,(n\,|\,\underset{\gamma}{\bar m}\,|\,\underset{\delta}{l})}}. \tag{11}$$

Operating on this equation by the *Law of Extensible Minors* we get

$$\left(A_{\substack{(\bar n\,|\,\underset{\alpha}{m})\,(n\,|\,\underset{\alpha}{m}\,|\,\underset{\beta}{l}\,|\,\underset{1}{k}) \\ (\bar n\,|\,\underset{\gamma}{m})\,(n\,|\,\underset{\gamma}{m}\,|\,\underset{\delta}{l}\,|\,\underset{1}{k})}} A_{\substack{(\bar n\,|\,\underset{\alpha}{m})\,(n\,|\,\underset{\alpha}{m}\,|\,\underset{\beta}{l}\,|\,\underset{2}{k}) \\ (\bar n\,|\,\underset{\gamma}{m})\,(n\,|\,\underset{\gamma}{m}\,|\,\underset{\delta}{l}\,|\,\underset{2}{k})}} \cdots\cdots A_{\substack{(\bar n\,|\,\underset{\alpha}{m})\,(n\,|\,\underset{\alpha}{m}\,|\,\underset{\beta}{l}\,|\,\underset{\sigma}{k}) \\ (\bar n\,|\,\underset{\gamma}{m})\,(n\,|\,\underset{\gamma}{m}\,|\,\underset{\delta}{l}\,|\,\underset{\sigma}{k})}}\right)$$

$$= A^{(l-1)_k}_{\substack{(\bar n\,|\,\underset{\alpha}{m}) \\ (\bar n\,|\,\underset{\gamma}{m})}} A^{(l-1)_k-1}_{\substack{(\bar n\,|\,\underset{\alpha}{m})\,(n\,|\,\underset{\alpha}{m}\,|\,\underset{\beta}{l}) \\ (\bar n\,|\,\underset{\gamma}{m})\,(n\,|\,\underset{\gamma}{m}\,|\,\underset{\delta}{l})}} \tag{12}$$

and

$$\left(A_{\substack{(n\,|\,\underset{\alpha}{\bar m}\,|\,\underset{\beta}{l})\,(n\,|\,\underset{\alpha}{m}\,|\,\underset{\beta}{l}\,|\,\underset{1}{k}) \\ (n\,|\,\underset{\gamma}{\bar m}\,|\,\underset{\delta}{l})\,(n\,|\,\underset{\gamma}{m}\,|\,\underset{\delta}{l}\,|\,\underset{1}{k})}} A_{\substack{(n\,|\,\underset{\alpha}{\bar m}\,|\,\underset{\beta}{l})\,(n\,|\,\underset{\alpha}{m}\,|\,\underset{\beta}{l}\,|\,\underset{2}{k}) \\ (n\,|\,\underset{\gamma}{\bar m}\,|\,\underset{\delta}{l})\,(n\,|\,\underset{\gamma}{m}\,|\,\underset{\delta}{l}\,|\,\underset{2}{k})}} \cdots\cdots A_{\substack{(n\,|\,\underset{\alpha}{\bar m}\,|\,\underset{\beta}{l})\,(n\,|\,\underset{\alpha}{m}\,|\,\underset{\beta}{l}\,|\,\underset{\sigma}{k}) \\ (n\,|\,\underset{\gamma}{\bar m}\,|\,\underset{\delta}{l})\,(n\,|\,\underset{\gamma}{m}\,|\,\underset{\delta}{l}\,|\,\underset{\sigma}{k})}}\right)$$

$$= A^{(l-1)_k}_{\substack{(n\,|\,\underset{\alpha}{\bar m}\,|\,\underset{\beta}{l}) \\ (n\,|\,\underset{\gamma}{\bar m}\,|\,\underset{\delta}{l})}} A^{(l-1)_k-1}_{\substack{(n\,|\,\underset{\alpha}{m}) \\ (n\,|\,\underset{\gamma}{m})}} \tag{13}$$

and

$$\left(A_{\substack{(n\,|\,\underset{\alpha}{m}\,|\,\underset{\beta}{l}\,|\,\underset{1}{k}) \\ (n\,|\,\underset{\gamma}{m}\,|\,\underset{\delta}{l}\,|\,\underset{1}{k})}} A_{\substack{(n\,|\,\underset{\alpha}{m}\,|\,\underset{\beta}{l}\,|\,\underset{2}{k}) \\ (n\,|\,\underset{\gamma}{m}\,|\,\underset{\delta}{l}\,|\,\underset{2}{k})}} \cdots\cdots A_{\substack{(n\,|\,\underset{\alpha}{m}\,|\,\underset{\beta}{l}\,|\,\underset{\sigma}{k}) \\ (n\,|\,\underset{\gamma}{m}\,|\,\underset{\delta}{l}\,|\,\underset{\sigma}{k})}}\right).$$

$$= A^{(l-1)_k}\ A^{(l-1)_k-1}_{\substack{(n\,|\,\underset{\alpha}{m}\,|\,\underset{\beta}{l}) \\ (n\,|\,\underset{\gamma}{m}\,|\,\underset{\delta}{l})}}. \tag{14}$$

14. If the constituents in the intersection of the last r columns and the last $(n-r)$ rows are zero, then $A=(-1)^{r(n-r)}A_{\substack{(n\,|\,\underset{1}{r}) \\ (\bar n\,|\,\underset{1}{n-r})}}\ A_{\substack{(\bar n\,|\,\underset{1}{r}) \\ (n\,|\,\underset{1}{n-r})}}$ and equation (8) becomes

$$\left(A_{\substack{(n\,|\,\underset{\alpha}{m}\,|\,\underset{1}{l}) \\ (n\,|\,\underset{\gamma}{m}\,|\,\underset{1}{l})}} A_{\substack{(n\,|\,\underset{\alpha}{m}\,|\,\underset{2}{l}) \\ (n\,|\,\underset{\gamma}{m}\,|\,\underset{2}{l})}} \cdots\cdots A_{\substack{(n\,|\,\underset{\alpha}{m}\,|\,\underset{\lambda}{l}) \\ (n\,|\,\underset{\gamma}{m}\,|\,\underset{\lambda}{l})}}\right)$$

$$= (-1)^{r(n-r)(m-1)_l}\, A^{(m-1)_l}_{\substack{(n\,|\,\underset{1}{r}) \\ (\bar n\,|\,\underset{1}{n-r})}}\ A^{(m-1)_l}_{\substack{(\bar n\,|\,\underset{1}{r}) \\ (n\,|\,\underset{1}{n-r})}}\ A^{(m-1)_l-1}_{\substack{(n\,|\,\underset{\alpha}{m}) \\ (n\,|\,\underset{\gamma}{m})}}. \tag{15}$$

If $n - r = r$, then equation (15) becomes

$$\left(A_{\substack{(n|m|l)\\ \alpha\;\,1\\ (n|m|l)\\ \gamma\;\,1}} A_{\substack{(n|m|l)\\ \alpha\;\,2\\ (n|m|l)\\ \gamma\;\,2}} \cdots\cdots A_{\substack{(n|m|l)\\ \alpha\;\,\lambda\\ (n|m|l)\\ \gamma\;\,\lambda}}\right)$$
$$= (-1)^{r^2(m-1)_l} A^{(m-1)_l}_{\substack{(n|r)\\ 1\\ (n|n-r)\\ 1}} \; A_{\substack{(\bar{n}|r)\\ 1\\ (n|n-r)\\ 1}} \; A^{(m-1)_{l-1}}_{\substack{(n|m)\\ \alpha\\ (n|m)\\ \gamma}}.$$ [*]

Similarly for equations (10) and (14).

15. The determinant

$$\left(A_{\substack{(n|m|l|k)\\ \alpha\;\,\beta\;\,1\\ (n|m|l|k)\\ \gamma\;\,\delta\;\,1}} A_{\substack{(n|m|l|k)\\ \alpha\;\,\beta\;\,2\\ (n|m|l|k)\\ \gamma\;\,\delta\;\,2}} \cdots\cdots A_{\substack{(n|m|l|k)\\ \alpha\;\,\beta\;\,\sigma\\ (n|m|l|k)\\ \gamma\;\,\delta\;\,\sigma}}\right)$$

is a minor of order σ of

$$\left(A_{\substack{(n|m|k)\\ \alpha\;\,1\\ (n|m|k)\\ \gamma\;\,1}} A_{\substack{(n|m|k)\\ \alpha\;\,2\\ (n|m|k)\\ \gamma\;\,2}} \cdots\cdots A_{\substack{(n|m|k)\\ \alpha\;\,\rho\\ (n|m|k)\\ \gamma\;\,\rho}}\right)_{\rho = m_k}$$

and by equation (14) is equal to

$$A^{(l-1)_k} A^{(l-1)_{k-1}}_{\substack{(n|m|l)\\ \alpha\;\,\beta\\ (n|m|l)\\ \gamma\;\,\delta}}.$$ [†]

16. The determinant

$$\left(A_{\substack{(n|m|k)\\ \alpha\;\,1\\ (n|m|k)\\ \gamma\;\,1}} A_{\substack{(n|m|k)\\ \alpha\;\,2\\ (n|m|k)\\ \gamma\;\,2}} \cdots\cdots A_{\substack{(n|m|k)\\ \alpha\;\,\rho\\ (n|m|k)\\ \gamma\;\,\rho}}\right)$$
$$= A^{(m-1)_k} A^{(m-1)_{k-1}}_{\substack{(n|m)\\ \alpha\\ (n|m)\\ \gamma}},$$

and the determinant

$$\left(A_{\substack{(n|\overline{m}|k)\\ \alpha\;\,1\\ (n|\overline{m}|k)\\ \gamma\;\,1}} A_{\substack{(n|\overline{m}|k)\\ \alpha\;\,2\\ (n|\overline{m}|k)\\ \gamma\;\,2}} \cdots\cdots A_{\substack{(n|\overline{m}|k)\\ \alpha\;\,\rho\\ (n|\overline{m}|k)\\ \gamma\;\,\rho}}\right)$$
$$= A^{(m-1)_{k-1}} A^{(m-1)_k}_{\substack{(n|m)\\ \alpha\\ (n|m)\\ \gamma}} \quad \text{(Art. 12).}$$

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[*] Cf. Scott, Proc. Lon. Math. Soc., vol. XIV, §8.

[†] Cf. Scott, ibid. art. 6, p. 96.

The product

$$\left(A_{\substack{(n|\underset{\alpha}{m}|\underset{1}{k}) \\ (n|\underset{\gamma}{m}|\underset{1}{k})}} A_{\substack{(n|\underset{\alpha}{m}|\underset{2}{k}) \\ (n|\underset{\gamma}{m}|\underset{2}{k})}} \cdots\cdots A_{\substack{(n|\underset{\alpha}{m}|\underset{i}{k}) \\ (n|\underset{\gamma}{m}|\underset{i}{k})}}\right)_{i<\rho}$$

$$\times\left(A_{\substack{(n|\underset{\alpha}{\overline{m}}|\underset{1}{k}) \\ (n|\underset{\gamma}{\overline{m}}|\underset{1}{k})}} A_{\substack{(n|\underset{\alpha}{\overline{m}}|\underset{2}{k}) \\ (n|\underset{\gamma}{\overline{m}}|\underset{2}{l})}} \cdots\cdots A_{\substack{(n|\underset{\alpha}{\overline{m}}|\underset{\rho}{k}) \\ (n|\underset{\gamma}{\overline{m}}|\underset{\rho}{k})}}\right),$$

$$= A^{i} A^{i}_{\substack{(n|\underset{\alpha}{m}) \\ (n|\underset{\gamma}{m})}} \left(A_{\substack{(n|\underset{\alpha}{\overline{m}}|\underset{i+1}{k}) \\ (n|\underset{\gamma}{\overline{m}}|\underset{i+1}{k})}} A_{\substack{(n|\underset{\alpha}{\overline{m}}|\underset{i+2}{k}) \\ (n|\underset{\gamma}{\overline{m}}|\underset{i+2}{k})}} \cdots\cdots A_{\substack{(n|\underset{\alpha}{\overline{m}}|\underset{\rho}{k}) \\ (n|\underset{\gamma}{\overline{m}}|\underset{\rho}{k})}}\right),$$

$$\therefore \left(A_{\substack{(n|\underset{\alpha}{m}|\underset{1}{k}) \\ (n|\underset{\gamma}{m}|\underset{1}{k})}} A_{\substack{(n|\underset{\alpha}{m}|\underset{2}{k}) \\ (n|\underset{\gamma}{m}|\underset{2}{k})}} \cdots\cdots A_{\substack{(n|\underset{\alpha}{m}|\underset{i}{k}) \\ (n|\underset{\gamma}{m}|\underset{i}{k})}}\right)$$

$$= A^{i-(m-1)_{k-1}} A^{i-(m-1)_{k}}_{\substack{(n|\underset{\alpha}{m}) \\ (n|\underset{\gamma}{m})}} \left(A_{\substack{(n|\underset{\alpha}{\overline{m}}|\underset{i+1}{k}) \\ (n|\underset{\gamma}{\overline{m}}|\underset{i+1}{k})}} A_{\substack{(n|\underset{\alpha}{\overline{m}}|\underset{i+2}{k}) \\ (n|\underset{\gamma}{\overline{m}}|\underset{i+2}{k})}} \cdots\cdots A_{\substack{(n|\underset{\alpha}{\overline{m}}|\underset{\rho}{k}) \\ (\overline{n}|\underset{\gamma}{m}|\underset{\rho}{k})}}\right) \tag{17}$$

If A has a block of zero constituents, its value may be given as in art. 14.*

17. The product

$$\left(A_{\substack{(n|\underset{1}{m}|\underset{1}{m-1}) \\ (n|\underset{1}{m}|\underset{1}{m-1})}} A_{\substack{(n|\underset{1}{m}|\underset{1}{m-2})(\overline{n}|\underset{1}{n-1}) \\ (n|\underset{1}{m}|\underset{2}{m-1})}} A_{\substack{(n|\underset{1}{m}|\underset{1}{m-3})(\overline{n}|\underset{1}{n-2}) \\ (n|\underset{1}{m}|\underset{3}{m-1})}} \cdots\cdots A_{\substack{(\overline{n}|n-m+1) \\ (n|\underset{1}{m}|\underset{m}{m-1})}}\right)$$

$$\times\left(A_{\substack{(n|\underset{1}{\overline{m}}|\underset{1}{m-1}) \\ (n|\underset{1}{\overline{m}}|\underset{1}{m-1})}} A_{\substack{(n|\overline{m}|m-1) \\ (n|\underset{1}{\overline{m}}|\underset{2}{m-1})}} \cdots\cdots A_{\substack{(n|\underset{1}{\overline{m}}|\underset{m}{m-1}) \\ (n|\underset{1}{\overline{m}}|\underset{m}{m-1})}}\right)$$

$$= A^{n}.\ A_{\substack{(n|\underset{1}{m}) \\ (n|\underset{1}{m})}}\ A_{\substack{(n|\underset{1}{m-1})(\overline{n}|\underset{1}{n-1}) \\ (n|\underset{1}{m})}}\ A_{\substack{(n|\underset{1}{m-2})(\overline{n}|\underset{1}{n-2}) \\ (n|\underset{1}{m})}} \cdots\cdots A_{\substack{(\underset{1}{n}|1)(\overline{n}|n-m+1) \\ (n|\underset{1}{m})}}$$

the product of the constituents along the principal diagonal of the product. The truth of this is seen on observing that all the constituents on the lower left-hand side of the principal diagonal of the product are zero.

The determinant

$$\left(A_{\substack{(n|\underset{1}{\overline{m}}|\underset{1}{m-1}) \\ (n|\underset{1}{\overline{m}}|\underset{1}{m-1})}} A_{\substack{(n|\overline{m}|\underset{2}{m-1}) \\ (n|\overline{m}|\underset{2}{m-1})}} \cdots\cdots A_{\substack{(n|\underset{1}{\overline{m}}|\underset{m}{m-1}) \\ (n|\underset{1}{\overline{m}}|\underset{m}{m-1})}}\right),$$

* Cf. Scott, ibid. art. 9.

being a minor of order m of $A_{(n-1)}$ is equal to

$$A^{m-1}A_{\substack{(n\,|\,\underset{1}{m})\\(n\,|\,\underset{1}{m})}}\,.$$

Therefore

$$\left(A_{\substack{(n\,|\,\underset{1}{m}\,|\,m\underset{1}{-}1)\\(n\,|\,\underset{1}{m}\,|\,m\underset{1}{-}1)}}A_{\substack{(n\,|\,\underset{1}{m}\,|\,m\underset{1}{-}2)(\bar{n}\,|\,n\underset{1}{-}1)\\(n\,|\,\underset{1}{m}\,|\,m\underset{2}{-}1)}}\cdots\cdots A_{\substack{(\bar{n}\,|\,n-\underset{1}{m}+1)\\(n\,|\,\underset{1}{m}\,|\,m\underset{m}{-}1)}}\right)$$

$$=A\,.\,A_{\substack{(n\,|\,m\underset{1}{-}1)(\bar{n}\,|\,n\underset{1}{-}1)\\(n\,|\,\underset{1}{m})}}\,.\,A_{\substack{(n\,|\,m\underset{1}{-}2)(\bar{n}\,|\,n\underset{1}{-}2)\\(n\,|\,\underset{1}{m})}}\cdots\cdots A_{\substack{(n\,|\,\underset{1}{1})(\bar{n}\,|\,n-\underset{1}{m}+1)\\(n\,|\,\underset{1}{m})}}\,.$$

This is the theorem given by Muir in his "Theory of Determinants,'

18. The product

$$\left(A_{\substack{(n\,|\,\underset{1}{m}\,|\,m\underset{1}{-}1)\\(n\,|\,\underset{1}{m}\,|\,m\underset{1}{-}1)}}A_{\substack{(n\,|\,\underset{1}{m}\,|\,m\underset{1}{-}2)(\bar{n}\,|\,n\underset{1}{-}1)\\(n\,|\,\underset{1}{m}\,|\,m\underset{2}{-}1)}}\cdots\cdots A_{\substack{(n\,|\,\underset{1}{m}\,|\,m\underset{1}{-}l)(\bar{n}\,|\,n\underset{1}{-}l+1)\\(n\,|\,\underset{1}{m}\,|\,m\underset{l}{-}1)}}\right)$$

$$\times\left(A_{\substack{(n\,|\,\underset{1}{\bar{m}}\,|\,m\underset{1}{-}1)\\(n\,|\,\underset{1}{\bar{m}}\,|\,m\underset{1}{-}1)}}A_{\substack{(n\,|\,\underset{1}{\bar{m}}\,|\,m\underset{2}{-}1)\\(n\,|\,\underset{1}{\bar{m}}\,|\,m\underset{2}{-}1)}}\cdots\cdots A_{\substack{(n\,|\,\underset{1}{\bar{m}}\,|\,m\underset{m}{-}1)\\(n\,|\,\underset{1}{\bar{m}}\,|\,m\underset{m}{-}1)}}\right)$$

$$=A^{l}.\,A_{\substack{(n\,|\,\underset{1}{m})\\(n\,|\,\underset{1}{m})}}A_{\substack{(n\,|\,m\underset{1}{-}1)(\bar{n}\,|\,n\underset{1}{-}1)\\(n\,|\,\underset{1}{m})}}\cdots\cdots A_{\substack{(n\,|\,m\underset{1}{-}l+1)(\bar{n}\,|\,n\underset{1}{-}l+1)\\(n\,|\,\underset{1}{m})}}$$

$$\times\left(A_{\substack{(n\,|\,\underset{1}{\bar{m}}\,|\,m\underset{l+1}{-}1)\\(n\,|\,\underset{1}{\bar{m}}\,|\,m\underset{l+1}{-}1)}}A_{\substack{(n\,|\,\underset{1}{\bar{m}}\,|\,m\underset{l+2}{-}1)\\(n\,|\,\underset{1}{\bar{m}}\,|\,m\underset{l+2}{-}1)}}\cdots\cdots A_{\substack{(n\,|\,\underset{1}{\bar{m}}\,|\,m\underset{m}{-}1)\\(n\,|\,\underset{1}{\bar{m}}\,|\,m\underset{m}{-}1)}}\right)$$

$$=A^{l}.\,A_{\substack{(n\,|\,\underset{1}{m})\\(n\,|\,\underset{1}{m})}}A_{\substack{(n\,|\,m\underset{1}{-}1)(\bar{n}\,|\,n\underset{1}{-}1)\\(n\,|\,\underset{1}{m})}}\cdots\cdots A_{\substack{(n\,|\,\underset{1}{\bar{m}}\,|\,m\underset{m}{-}1)\\(n\,|\,\underset{1}{\bar{m}}\,|\,m\underset{m}{-}1)}}$$

$$\times\,A^{m-l-1}.\,A_{\substack{(n\,|\,m\underset{1}{-}l)\\(n\,|\,m\underset{1}{-}l)}}$$

$$=A^{m-1}A_{\substack{(n\,|\,\underset{1}{m})\\(n\,|\,\underset{1}{m})}}A_{\substack{(n\,|\,m\underset{1}{-}l)\\(n\,|\,m\underset{1}{-}l)}}A_{\substack{(n\,|\,m\underset{1}{-}1)(\bar{n}\,|\,n\underset{1}{-}1)\\(n\,|\,\underset{1}{m})}}\cdots\cdots A_{\substack{(n\,|\,m\underset{1}{-}l+1)(\bar{n}\,|\,n\underset{1}{-}l+1\\(n\,|\,\underset{1}{m})}}$$

Therefore

$$\left(A_{\substack{(n\,|\,\underset{1}{m}\,|\,m\underset{1}{-}1)\\(n\,|\,\underset{1}{m}\,|\,m\underset{1}{-}1)}}A_{\substack{(n\,|\,\underset{1}{m}\,|\,m\underset{1}{-}2)(\bar{n}\,|\,n\underset{1}{-}1)\\(n\,|\,\underset{1}{m}\,|\,m\underset{2}{-}1)}}\cdots\cdots A_{\substack{(n\,|\,m\,|\,m\underset{1}{-}l).(\bar{n}\,|\,n\underset{1}{-}l+1)\\(n\,|\,\underset{1}{m}\,|\,m\underset{l}{-}1)}}\right)$$

$$=A_{\substack{(n\,|\,m\underset{1}{-}l)\\(n\,|\,m\underset{1}{-}l)}}A_{\substack{(n\,|\,\underset{1}{m}-1)(\bar{n}\,|\,n\underset{1}{-}1)\\(n\,|\,\underset{1}{m})}}\cdots\cdots A_{\substack{(n\,|\,m\underset{1}{-}l+1).(\bar{n}\,|\,n\underset{1}{-}l+1)\\(n\,|\,\underset{1}{m})}}\,.$$

Operating on equation (18) by the *Law of Complementaries* we get

$$
\begin{aligned}
&\left(A_{\substack{(\overline{n}\,|\,\underset{1}{m-1}) \\ (\overline{n}\,|\,\underset{1}{m-1})}}\; A_{\substack{(n\,|\,\underset{1}{\overline{n-1}}\,|\,\underset{1}{m-2}) \\ (\overline{n}\,|\,\underset{1}{m})(n\,|\,\underset{1}{\overline{m-1}}\,|\,\underset{1}{m-2})}}\; A_{\substack{(n\,|\,\underset{1}{\overline{n-2}}\,|\,\underset{1}{m-3}) \\ (\overline{n}\,|\,\underset{1}{m})(n\,|\,\underset{1}{\overline{m-2}}\,|\,\underset{1}{m-3})}} \cdots\cdots A_{\substack{(n\,|\,\underset{1}{n-m+1}) \\ (\overline{n}\,|\,\underset{1}{m})(n\,|\,\underset{1}{1})}}\right) \\
&= A_{\substack{(n\,|\,\underset{1}{\overline{n-1}}\,|\,\underset{1}{m-1}) \\ (\overline{n}\,|\,\underset{1}{m})}} A_{\substack{(n\,|\,\underset{1}{\overline{n-2}}\,|\,\underset{1}{m-2}) \\ (\overline{n}\,|\,\underset{1}{m})}} \cdots\cdots A_{\substack{(n\,|\,\underset{1}{\overline{n-m+1}}\,|\,\underset{1}{1}) \\ (\overline{n}\,|\,\underset{1}{m})}}.
\end{aligned}
\tag{20}
$$

19. The determinant obtained on writing $(n\,|\,\underset{\alpha}{m}\,|\,\underset{\gamma}{l}\,|\,\underset{\delta}{k})(n\,|\,\underset{\alpha}{\overline{m}}\,|\,\underset{\gamma}{l}\,|\,\underset{\epsilon}{l-k})$ for $(n\,|\,\underset{\alpha}{m}\,|\,\underset{\gamma}{l})$ in

$$\left(A_{\substack{(n\,|\,\underset{\alpha}{m}\,|\,\underset{1}{l}) \\ (n\,|\,\underset{\beta}{m}\,|\,\underset{1}{l})}} A_{\substack{(n\,|\,\underset{\alpha}{m}\,|\,\underset{2}{l}) \\ (n\,|\,\underset{\rho}{m}\,|\,\underset{2}{l})}} \cdots\cdots A_{\substack{(n\,|\,\underset{\alpha}{m}\,|\,\underset{\lambda}{l}) \\ (n\,|\,\underset{\beta}{m}\,|\,\underset{\lambda}{l})}}\right)$$

evidently has two rows identical and therefore vanishes.

20. The determinant M_1, obtained on writing $(n\,|\,\underset{\alpha}{m}\,|\,\underset{\gamma}{l}\,|\,\underset{\delta}{k})(\overline{n}\,|\,\underset{\alpha}{m}\,|\,\underset{\epsilon}{l-k})$ for $(n\,|\,\underset{\alpha}{m}\,|\,\underset{\gamma}{l})$* in

$$\left(A_{\substack{(n\,|\,\underset{\alpha}{m}\,|\,\underset{1}{l}) \\ (n\,|\,\underset{\beta}{m}\,|\,\underset{1}{l})}} A_{\substack{(n\,|\,\underset{\alpha}{m}\,|\,\underset{2}{l}) \\ (n\,|\,\underset{\beta}{m}\,|\,\underset{2}{l})}} \cdots\cdots A_{\substack{(n\,|\,\underset{\alpha}{m}\,|\,\underset{\lambda}{l}) \\ (n\,|\,\underset{\beta}{m}\,|\,\underset{\lambda}{l})}}\right)$$

vanishes, though two rows are not identical. For if we multiply M_1 by

$$\left(A_{\substack{(n\,|\,\underset{\alpha}{\overline{m}}\,|\,\underset{1}{l}) \\ (n\,|\,\underset{\beta}{\overline{m}}\,|\,\underset{1}{l})}} A_{\substack{(n\,|\,\underset{\alpha}{\overline{m}}\,|\,\underset{2}{l}) \\ (n\,|\,\underset{\gamma}{\overline{m}}\,|\,\underset{2}{l})}} \cdots\cdots A_{\substack{(n\,|\,\underset{\alpha}{\overline{m}}\,|\,\underset{\lambda}{l}) \\ (n\,|\,\underset{\beta}{\overline{m}}\,|\,\underset{\lambda}{l})}}\right),$$

every constituent in the γ^{th} column of the product will be zero since the upper line of every suffix of M_1 contains some number in common with the combination $(n\,|\,\underset{\alpha}{\overline{m}}\,|\,\underset{\gamma}{l})$. The product therefore vanishes, and since the multiplier is in general different from zero, M_1 must vanish.

21. Any determinant M_2 of order λ, the upper line of every suffix of which contains at least one of some $(m-l)$ or fewer numbers and the lower lines of the suffixes are the combinations l at a time of some m of the n numbers, vanishes.

This theorem, which is perhaps a little more comprehensive than that of the last article, is proven in a similar way.

---

* We may, of course, have similar substitutions in the upper lines of the suffixes of the constituents in other rows at the same time.

Let the numbers in the combination $(n|\overline{m}|l)_{\alpha\,\gamma}$ be the $(m-l)$ numbers in question, and let the lower lines of the suffixes be the combinations $(n|m|l)_{\beta\,1}$, $(n|m|l)_{\beta\,2} \dots\dots (n|m|l)_{\beta\,\lambda}$.

If we multiply M_2 by

$$\left(A_{\substack{(n|\overline{m}|l)_{\alpha\,1}\\(n|\overline{m}|l)_{\beta\,1}}} A_{\substack{(n|\overline{m}|l)_{\alpha\,2}\\(n|\overline{m}|l)_{\beta\,2}}} \dots\dots A_{\substack{(n|\overline{m}|l)_{\alpha\,\lambda}\\(n|\overline{m}|l)_{\beta\,\lambda}}}\right),$$

every constituent in the γ^{th} column of the product will be zero, since the upper line of every constituent of M_2 contains some number in common with the combination $(n|\overline{m}|l)_{\alpha\,\gamma}$. Therefore the product and consequently M_2 vanishes.

22. Every minor of the determinant M_2 of the last article, which is of order $\overline{\gtrless}\{(m-k)_l+1\}$ and which contains any k numbers in at least $\{(m-k)_l+1\}$ of the upper lines of the suffixes, vanishes.

Without loss of generality we may suppose that the upper lines of the suffixes of the constituents in the first $\{(m-k)_l+1\}$ rows of M_2 contain the same k numbers.

Multiplying any minor containing these rows by the same multiplier as in the last article, it is readily seen that every constituent in the intersection of the first $\{(m-k)_l+1\}$ rows and last $\{m_l-(m-k)_l\}$ columns of the product are zero. Therefore the product and consequently the minor vanishes.

23. Any minor of

$$\left(A_{\substack{(n|m|l)_{\alpha\,1}\\(n|m|l)_{\beta\,1}}} A_{\substack{(n|m|l)_{\alpha\,2}\\(n|m|l)_{\beta\,2}}} \dots\dots A_{\substack{(n|m|l)_{\alpha\,\lambda}\\(n|m|l)_{\beta\,\lambda}}}\right)$$

can be expressed as a product of A and its minors whenever the complementary of the corresponding minor of

$$\left(A_{\substack{(n|\overline{m}|l)_{\alpha\,1}\\(n|\overline{m}|l)_{\beta\,1}}} A_{\substack{(n|\overline{m}|l)_{\alpha\,2}\\(n|\overline{m}|l)_{\beta\,2}}} \dots\dots A_{\substack{(n|\overline{m}|l)_{\alpha\,\lambda}\\(n|\overline{m}|l)_{\beta\,\lambda}}}\right)$$

can be so expressed.

If we know the value of any minor of $A_{(m)}$, the *Law of Complementaries* gives us the value of the corresponding minor of $A_{(n-m)}$, and the theorem of art. 1

gives us the value of the complementary of the corresponding minor of $A_{(n-m)}$, which again operated on by the *Law of Complementaries* gives the value of the complementary of the original minor of $A_{(m)}$. To determine the value of all the minors of $A_{(m)}$ and $A_{(n-m)}$ all that is necessary therefore is to know the value of the minors of any $\left\{\frac{n_m}{2}\right\}$* orders of one of them.

24. Let us illustrate the foregoing principles by finding the value of minors of $A_{(2)}$, the second compound of the determinant A of order five.

The product

$$\begin{vmatrix} A_{\substack{12\\12}} & A_{\substack{12\\13}} & A_{\substack{12\\23}} \\ A_{\substack{13\\12}} & A_{\substack{13\\13}} & A_{\substack{13\\23}} \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} A_{33} & -A_{32} & A_{31} \\ -A_{23} & A_{22} & -A_{21} \\ A_{13} & -A_{12} & A_{11} \end{vmatrix} = \begin{vmatrix} A\,.\,A_{\substack{123\\123}} & 0 & 0 \\ 0 & A\,.\,A_{\substack{123\\123}} & 0 \\ A_{31} & -A_{21} & A_{11} \end{vmatrix} = A^2 A_{11} A^2_{\substack{123\\123}}.$$

$$\therefore \left(A_{\substack{12\\12}} A_{\substack{13\\13}}\right) = A\,.\,A_{\substack{123\\123}}.$$

The product

$$\begin{vmatrix} A_{\substack{12\\12}} & A_{\substack{12\\13}} & A_{\substack{12\\14}} & A_{\substack{12\\15}} \\ A_{\substack{13\\12}} & A_{\substack{13\\13}} & A_{\substack{13\\14}} & A_{\substack{13\\15}} \\ A_{\substack{14\\12}} & A_{\substack{14\\13}} & A_{\substack{14\\14}} & A_{\substack{14\\15}} \\ 0 & 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} a_{22} & -a_{23} & a_{24} & -a_{25} \\ -a_{32} & a_{33} & -a_{34} & a_{35} \\ a_{42} & -a_{43} & a_{44} & -a_{45} \\ -a_{52} & a_{53} & -a_{54} & a_{55} \end{vmatrix}$$

$$= \begin{vmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{11} & 0 & 0 \\ 0 & 0 & A_{11} & 0 \\ -a_{25} & a_{35} & -a_{45} & a_{55} \end{vmatrix} = a_{55} A_{11}^3,$$

$$\therefore \left(A_{\substack{12\\12}} A_{\substack{13\\13}} A_{\substack{14\\14}}\right) = a_{55} A_{11}^3 \text{ or } A_{11}^3 A_{\substack{1234\\1234}}.$$

The product

$$\begin{vmatrix} A_{\substack{12\\12}} & A_{\substack{12\\13}} & A_{\substack{12\\23}} \\ A_{\substack{13\\12}} & A_{\substack{13\\13}} & A_{\substack{13\\23}} \\ A_{\substack{23\\12}} & A_{\substack{23\\13}} & A_{\substack{23\\23}} \end{vmatrix} \cdot \begin{vmatrix} A_{33} & -A_{32} & A_{31} \\ -A_{23} & A_{22} & -A_{21} \\ A_{13} & -A_{12} & A_{11} \end{vmatrix}$$

---

* $\left\{\frac{n_m}{2}\right\}$ denotes the greatest integer in $\frac{n_m}{2}$.

$$=\begin{vmatrix} A\,.\,A_{\substack{123\\123}} & 0 & 0 \\ 0 & A\,.\,A_{\substack{123\\123}} & 0 \\ 0 & 0 & A\,.\,A_{\substack{123\\123}} \end{vmatrix} = A^3.\,A^3_{\substack{123\\123}},$$

$$\therefore \left(A_{\substack{12\\12}}\,A_{\substack{13\\13}}\,A_{\substack{23\\23}}\right) = A\,.\,A^2_{\substack{123\\123}} \quad \text{(Eq. (8))}.$$

The product (by columns)

$$\begin{vmatrix} A_{\substack{12\\12}} & A_{\substack{12\\15}} & A_{\substack{12\\45}} \\ A_{\substack{13\\12}} & A_{\substack{13\\15}} & A_{\substack{13\\45}} \\ A_{\substack{23\\12}} & A_{\substack{23\\15}} & A_{\substack{23\\45}} \end{vmatrix} . \begin{vmatrix} A_{33} & -A_{32} & A_{31} \\ -A_{23} & A_{22} & -A_{21} \\ A_{13} & -A_{12} & A_{11} \end{vmatrix}$$

$$=\begin{vmatrix} A\,.\,A_{\substack{123\\123}} & 0 & 0 \\ -A\,.\,A_{\substack{123\\135}} & A\,.\,A_{\substack{123\\125}} & 0 \\ A\,.\,A_{\substack{123\\345}} & A\,.\,A_{\substack{123\\245}} & A\,.\,A_{\substack{123\\145}} \end{vmatrix} = A^3.\,A_{\substack{123\\123}}\,A_{\substack{123\\125}}\,A_{\substack{123\\145}},$$

$$\therefore \left(A_{\substack{12\\12}}\,A_{\substack{13\\15}}\,A_{\substack{23\\45}}\right) = A\,.\,A_{\substack{123\\125}}\,.\,A_{\substack{123\\145}} \quad \text{(Eq. (18))}.$$

The product

$$\begin{vmatrix} A_{\substack{12\\12}} & A_{\substack{12\\13}} & A_{\substack{12\\23}} \\ A_{\substack{13\\12}} & A_{\substack{13\\13}} & A_{\substack{13\\23}} \\ A_{\substack{25\\12}} & A_{\substack{25\\13}} & A_{\substack{25\\23}} \end{vmatrix} . \begin{vmatrix} A_{33} & -A_{32} & A_{31} \\ -A_{23} & A_{22} & -A_{21} \\ A_{13} & -A_{12} & A_{11} \end{vmatrix}$$

$$=\begin{vmatrix} A\,.\,A_{\substack{123\\123}} & 0 & 0 \\ 0 & A\,.\,A_{\substack{123\\123}} & 0 \\ -A\,.\,A_{\substack{235\\123}} & 0 & A\,.\,A_{\substack{125\\123}} \end{vmatrix} = A^3.\,A^2_{\substack{123\\123}}\,A_{\substack{125\\123}},$$

$$\therefore \left(A_{\substack{12\\12}}\,A_{\substack{13\\13}}\,A_{\substack{25\\23}}\right) = A\,.\,A_{\substack{123\\123}}\,A_{\substack{125\\123}}.$$

The product

$$\begin{vmatrix} A_{\substack{12\\12}} & A_{\substack{12\\13}} & A_{\substack{12\\14}} & A_{\substack{12\\23}} & A_{\substack{12\\24}} & A_{\substack{12\\34}} \\ A_{\substack{13\\12}} & A_{\substack{13\\13}} & A_{\substack{13\\14}} & A_{\substack{13\\23}} & A_{\substack{13\\24}} & A_{\substack{13\\34}} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ A_{\substack{24\\12}} & A_{\substack{24\\13}} & A_{\substack{24\\14}} & A_{\substack{24\\23}} & A_{\substack{24\\24}} & A_{\substack{24\\34}} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} A_{\substack{34\\34}} & -A_{\substack{34\\24}} & A_{\substack{34\\23}} & A_{\substack{34\\14}} & -A_{\substack{34\\13}} & A_{\substack{34\\12}} \\ -A_{\substack{24\\34}} & A_{\substack{24\\24}} & -A_{\substack{24\\23}} & -A_{\substack{24\\14}} & A_{\substack{24\\13}} & -A_{\substack{24\\12}} \\ A_{\substack{23\\34}} & -A_{\substack{23\\24}} & A_{\substack{23\\23}} & A_{\substack{23\\14}} & -A_{\substack{23\\13}} & A_{\substack{23\\12}} \\ A_{\substack{14\\34}} & -A_{\substack{14\\24}} & A_{\substack{14\\23}} & A_{\substack{14\\14}} & -A_{\substack{14\\13}} & A_{\substack{14\\12}} \\ -A_{\substack{13\\34}} & A_{\substack{13\\24}} & -A_{\substack{13\\23}} & -A_{\substack{13\\14}} & A_{\substack{13\\13}} & -A_{\substack{13\\12}} \\ A_{\substack{12\\34}} & -A_{\substack{12\\24}} & A_{\substack{12\\23}} & A_{\substack{12\\14}} & -A_{\substack{12\\13}} & A_{\substack{12\\12}} \end{vmatrix}$$

$$= \begin{vmatrix} A\,.\,a_{55} & 0 & 0 & 0 & 0 & 0 \\ 0 & A\,.\,a_{55} & 0 & 0 & 0 & 0 \\ A_{\substack{34\\23}} & -A_{\substack{24\\23}} & A_{\substack{23\\23}} & A_{\substack{14\\23}} & -A_{\substack{13\\23}} & A_{\substack{12\\23}} \\ A_{\substack{34\\14}} & -A_{\substack{24\\14}} & A_{\substack{23\\14}} & A_{\substack{14\\14}} & -A_{\substack{13\\14}} & A_{\substack{12\\14}} \\ 0 & 0 & 0 & 0 & A\,.\,a_{55} & 0 \\ A_{\substack{34\\12}} & -A_{\substack{24\\12}} & A_{\substack{23\\12}} & A_{\substack{14\\12}} & -A_{\substack{13\\12}} & A_{\substack{12\\12}} \end{vmatrix} = A^3\,.\,a_{55}^3 \left(A_{\substack{12\\12}} A_{\substack{14\\14}} A_{\substack{23\\23}} \right),$$

$$\therefore \left(A_{\substack{12\\12}} A_{\substack{13\\13}} A_{\substack{24\\24}} \right) = \left(A_{\substack{12\\12}} A_{\substack{14\\14}} A_{\substack{23\\23}} \right).$$

The product

$$\begin{vmatrix} A_{\substack{12\\12}} & A_{\substack{12\\13}} & A_{\substack{12\\23}} \\ A_{\substack{13\\12}} & A_{\substack{13\\13}} & A_{\substack{13\\23}} \\ A_{\substack{14\\12}} & A_{\substack{14\\13}} & A_{\substack{14\\23}} \end{vmatrix} \cdot \begin{vmatrix} A_{33} & A_{32} & A_{31} \\ A_{23} & A_{22} & A_{21} \\ A_{13} & A_{12} & A_{11} \end{vmatrix} = \begin{vmatrix} A\,.\,A_{\substack{123\\123}} & 0 & 0 \\ 0 & A\,.\,A_{\substack{123\\123}} & 0 \\ -A\,.\,A_{\substack{124\\123}} & A\,.\,A_{\substack{124\\123}} & 0 \end{vmatrix} = 0,$$

$$\therefore \left(A_{\substack{12\\12}} A_{\substack{13\\13}} A_{\substack{14\\23}} \right) = 0 \quad \text{(arts. 20, 21).}$$

There are $2 \times 10 \times 20 = 400$ vanishing minors of order three.

The product

$$\begin{vmatrix} A_{\substack{12\\12}} & A_{\substack{12\\13}} & A_{\substack{12\\14}} & A_{\substack{12\\15}} \\ A_{\substack{13\\12}} & A_{\substack{13\\13}} & A_{\substack{13\\14}} & A_{\substack{13\\15}} \\ A_{\substack{14\\12}} & A_{\substack{14\\13}} & A_{\substack{14\\14}} & A_{\substack{14\\15}} \\ A_{\substack{15\\12}} & A_{\substack{15\\13}} & A_{\substack{15\\14}} & A_{\substack{15\\15}} \end{vmatrix} \cdot \begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ a_{32} & a_{33} & a_{34} & a_{35} \\ a_{42} & a_{43} & a_{44} & a_{45} \\ a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} = \begin{vmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{11} & 0 & 0 \\ 0 & 0 & A_{11} & 0 \\ 0 & 0 & 0 & A_{11} \end{vmatrix} = A_{11}^4,$$

$$\therefore \left(A_{\substack{12\\12}} A_{\substack{13\\13}} A_{\substack{14\\14}} A_{\substack{15\\15}} \right) = A_{11}^3 \quad \text{(Eq. (7)).}$$

The product

$$\begin{vmatrix} A_{\substack{12\\12}} & A_{\substack{12\\13}} & A_{\substack{12\\14}} & A_{\substack{12\\23}} & A_{\substack{12\\24}} & A_{\substack{12\\34}} \\ A_{\substack{13\\12}} & A_{\substack{13\\13}} & A_{\substack{13\\14}} & A_{\substack{13\\23}} & A_{\substack{13\\24}} & A_{\substack{13\\34}} \\ A_{\substack{14\\12}} & A_{\substack{14\\13}} & A_{\substack{14\\14}} & A_{\substack{14\\23}} & A_{\substack{14\\24}} & A_{\substack{14\\34}} \\ A_{\substack{23\\12}} & A_{\substack{23\\13}} & A_{\substack{23\\14}} & A_{\substack{23\\23}} & A_{\substack{23\\24}} & A_{\substack{23\\34}} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} A_{\substack{34\\34}} & -A_{\substack{34\\24}} & A_{\substack{34\\23}} & A_{\substack{34\\14}} & -A_{\substack{34\\13}} & A_{\substack{34\\12}} \\ -A_{\substack{24\\34}} & A_{\substack{24\\24}} & -A_{\substack{24\\23}} & -A_{\substack{24\\14}} & A_{\substack{24\\13}} & -A_{\substack{24\\12}} \\ A_{\substack{23\\34}} & -A_{\substack{23\\24}} & A_{\substack{23\\23}} & A_{\substack{23\\14}} & -A_{\substack{23\\13}} & A_{\substack{23\\12}} \\ A_{\substack{14\\34}} & -A_{\substack{14\\24}} & A_{\substack{14\\23}} & A_{\substack{14\\14}} & -A_{\substack{14\\13}} & A_{\substack{14\\12}} \\ -A_{\substack{13\\34}} & A_{\substack{13\\24}} & -A_{\substack{13\\23}} & -A_{\substack{13\\14}} & A_{\substack{13\\13}} & -A_{\substack{13\\12}} \\ A_{\substack{12\\34}} & -A_{\substack{12\\24}} & A_{\substack{12\\23}} & A_{\substack{12\\14}} & -A_{\substack{12\\13}} & A_{\substack{12\\12}} \end{vmatrix}$$

$$\begin{vmatrix} A\,.\,a_{55} & 0 & 0 & 0 & 0 & 0 \\ 0 & A\,.\,a_{55} & 0 & 0 & 0 & 0 \\ 0 & 0 & A\,.\,a_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & A\,.\,a_{55} & 0 & 0 \\ -A_{\substack{34\\13}} & A_{\substack{24\\13}} & -A_{\substack{23\\13}} & -A_{\substack{14\\13}} & A_{\substack{13\\13}} & -A_{\substack{12\\13}} \\ A_{\substack{34\\12}} & -A_{\substack{24\\12}} & A_{\substack{23\\12}} & A_{\substack{14\\12}} & -A_{\substack{13\\12}} & A_{\substack{12\\12}} \end{vmatrix} = A^4\,.\,a_{55}^4 \begin{pmatrix} A_{\substack{12\\12}} & A_{\substack{13\\13}} \end{pmatrix}$$

$$= A^4\,.\,a_{55}^4\,A_{11}\,A_{\substack{123\\123}},$$

$$\therefore \begin{pmatrix} A_{\substack{12\\12}} & A_{\substack{13\\13}} & A_{\substack{14\\14}} & A_{\substack{23\\23}} \end{pmatrix} = A\,.\,A_{11}\,A_{\substack{123\\123}}\,A_{\substack{1234\\1234}}.$$

The product

$$\begin{vmatrix} A_{\substack{12\\12}} & A_{\substack{12\\13}} & A_{\substack{12\\14}} & A_{\substack{12\\15}} \\ A_{\substack{23\\12}} & A_{\substack{23\\13}} & A_{\substack{23\\14}} & A_{\substack{23\\15}} \\ A_{\substack{34\\12}} & A_{\substack{34\\13}} & A_{\substack{34\\14}} & A_{\substack{34\\15}} \\ A_{\substack{45\\12}} & A_{\substack{45\\13}} & A_{\substack{45\\14}} & A_{\substack{45\\15}} \end{vmatrix} \cdot \begin{vmatrix} a_{22} & -a_{23} & a_{24} & -a_{25} \\ -a_{32} & a_{33} & -a_{34} & a_{35} \\ a_{42} & -a_{43} & a_{44} & -a_{45} \\ -a_{52} & a_{53} & -a_{54} & a_{55} \end{vmatrix}$$

$$= \begin{vmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{21} & 0 & 0 \\ 0 & 0 & A_{31} & 0 \\ 0 & 0 & 0 & A_{41} \end{vmatrix} = A_{11}\,A_{21}\,A_{31}\,A_{41},$$

$$\therefore \begin{pmatrix} A_{\substack{12\\12}} & A_{\substack{23\\13}} & A_{\substack{34\\14}} & A_{\substack{45\\15}} \end{pmatrix} = A_{21}\,A_{31}\,A_{41} \quad \text{(Eq. (20))}.$$

The product

$$\begin{vmatrix} A_{\substack{12\\12}} & A_{\substack{12\\13}} & A_{\substack{12\\14}} & A_{\substack{12\\23}} & A_{\substack{12\\24}} & A_{\substack{12\\34}} \\ A_{\substack{13\\12}} & A_{\substack{13\\13}} & A_{\substack{13\\14}} & A_{\substack{13\\23}} & A_{\substack{13\\24}} & A_{\substack{13\\34}} \\ A_{\substack{14\\12}} & A_{\substack{14\\13}} & A_{\substack{14\\14}} & A_{\substack{14\\23}} & A_{\substack{14\\24}} & A_{\substack{14\\34}} \\ A_{\substack{15\\12}} & A_{\substack{15\\13}} & A_{\substack{15\\14}} & A_{\substack{15\\23}} & A_{\substack{15\\24}} & A_{\substack{15\\34}} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} A_{\substack{34\\34}} & -A_{\substack{34\\24}} & A_{\substack{34\\23}} & A_{\substack{34\\14}} & -A_{\substack{34\\13}} & A_{\substack{34\\12}} \\ -A_{\substack{24\\34}} & A_{\substack{24\\24}} & -A_{\substack{24\\23}} & -A_{\substack{24\\14}} & A_{\substack{24\\13}} & -A_{\substack{24\\12}} \\ A_{\substack{23\\34}} & -A_{\substack{23\\24}} & A_{\substack{23\\23}} & A_{\substack{23\\14}} & -A_{\substack{23\\13}} & A_{\substack{23\\12}} \\ A_{\substack{14\\34}} & -A_{\substack{14\\24}} & A_{\substack{14\\23}} & A_{\substack{14\\14}} & -A_{\substack{14\\13}} & A_{\substack{14\\12}} \\ -A_{\substack{13\\34}} & A_{\substack{13\\24}} & -A_{\substack{13\\23}} & -A_{\substack{13\\14}} & A_{\substack{13\\13}} & -A_{\substack{13\\12}} \\ A_{\substack{12\\34}} & -A_{\substack{12\\24}} & A_{\substack{12\\23}} & A_{\substack{12\\14}} & -A_{\substack{12\\13}} & A_{\substack{12\\12}} \end{vmatrix}$$

$$= \begin{vmatrix} A\,.\,a_{55} & 0 & 0 & 0 & 0 & 0 \\ 0 & A\,.\,a_{55} & 0 & 0 & 0 & 0 \\ 0 & 0 & A\,.\,a_{55} & 0 & 0 & 0 \\ A\,.\,a_{25} & A\,.\,a_{35} & A\,.\,a_{45} & 0 & 0 & 0 \\ -A_{\substack{34\\13}} & A_{\substack{24\\13}} & -A\,.\,a_{\substack{23\\13}} & -A_{\substack{14\\13}} & A_{\substack{13\\13}} & -A_{\substack{12\\13}} \\ A_{\substack{34\\12}} & -A_{\substack{24\\12}} & A_{\substack{23\\12}} & A_{\substack{14\\12}} & -A_{\substack{13\\12}} & A_{\substack{12\\12}} \end{vmatrix} = 0.$$

$$\therefore \left(A_{\substack{12\\12}}\, A_{\substack{13\\13}}\, A_{\substack{14\\14}}\, A_{\substack{15\\23}}\right) = 0 \quad \text{(art. 22)}.$$

From this it is apparent that any determinant of order four formed from the matrix

$$\begin{Vmatrix} A_{\substack{12\\12}} & A_{\substack{12\\13}} & A_{\substack{12\\14}} & A_{\substack{12\\23}} & A_{\substack{12\\24}} & A_{\substack{12\\34}} \\ A_{\substack{13\\12}} & A_{\substack{13\\13}} & A_{\substack{13\\14}} & A_{\substack{13\\23}} & A_{\substack{13\\24}} & A_{\substack{13\\34}} \\ A_{\substack{14\\12}} & A_{\substack{14\\13}} & A_{\substack{14\\14}} & A_{\substack{14\\23}} & A_{\substack{14\\24}} & A_{\substack{14\\34}} \\ A_{\substack{15\\12}} & A_{\substack{15\\13}} & A_{\substack{15\\14}} & A_{\substack{15\\23}} & A_{\substack{15\\24}} & A_{\substack{15\\34}} \end{Vmatrix}$$

vanishes.

There are $2.\{75 \times 5\} = 750$ vanishing minors of order four.

The product

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ A_{\substack{24\\23}} & A_{\substack{24\\24}} & A_{\substack{24\\25}} & A_{\substack{24\\34}} & A_{\substack{24\\35}} & A_{\substack{24\\45}} \\ A_{\substack{25\\23}} & A_{\substack{25\\24}} & A_{\substack{25\\25}} & A_{\substack{25\\34}} & A_{\substack{25\\35}} & A_{\substack{25\\45}} \\ A_{\substack{34\\23}} & A_{\substack{34\\24}} & A_{\substack{34\\25}} & A_{\substack{34\\34}} & A_{\substack{34\\35}} & A_{\substack{34\\45}} \\ A_{\substack{35\\23}} & A_{\substack{35\\24}} & A_{\substack{35\\25}} & A_{\substack{35\\34}} & A_{\substack{35\\35}} & A_{\substack{35\\45}} \\ A_{\substack{45\\23}} & A_{\substack{45\\24}} & A_{\substack{45\\25}} & A_{\substack{45\\34}} & A_{\substack{45\\35}} & A_{\substack{45\\45}} \end{vmatrix} \cdot \begin{vmatrix} A_{\substack{45\\45}} & -A_{\substack{45\\35}} & A_{\substack{45\\34}} & A_{\substack{45\\25}} & -A_{\substack{45\\24}} & A_{\substack{45\\23}} \\ -A_{\substack{35\\45}} & A_{\substack{35\\35}} & -A_{\substack{35\\34}} & -A_{\substack{35\\25}} & A_{\substack{35\\24}} & -A_{\substack{35\\23}} \\ A_{\substack{34\\45}} & -A_{\substack{34\\35}} & A_{\substack{34\\34}} & A_{\substack{34\\25}} & -A_{\substack{34\\24}} & A_{\substack{34\\23}} \\ A_{\substack{25\\45}} & -A_{\substack{25\\35}} & A_{\substack{25\\34}} & A_{\substack{25\\25}} & -A_{\substack{25\\24}} & A_{\substack{25\\23}} \\ -A_{\substack{24\\45}} & A_{\substack{24\\35}} & -A_{\substack{24\\34}} & -A_{\substack{24\\25}} & A_{\substack{24\\24}} & -A_{\substack{24\\23}} \\ A_{\substack{23\\45}} & -A_{\substack{23\\35}} & A_{\substack{23\\34}} & A_{\substack{23\\25}} & -A_{\substack{23\\24}} & A_{\substack{23\\23}} \end{vmatrix}$$

$$
= \begin{vmatrix}
A_{\substack{45\\45}} & -A_{\substack{35\\45}} & A_{\substack{34\\45}} & -A_{\substack{25\\45}} & A_{\substack{24\\45}} & -A_{\substack{23\\45}} \\
0 & A.a_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & A.a_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & A.a_{11} & 0 & 0 \\
0 & 0 & 0 & 0 & A.a_{11} & 0 \\
0 & 0 & 0 & 0 & 0 & A.a_{11}
\end{vmatrix} = a_{11}^5\, A^5\, A_{\substack{45\\45}},
$$

$$
\therefore \left(A_{\substack{24\\24}}\, A_{\substack{25\\25}}\, A_{\substack{34\\34}}\, A_{\substack{35\\35}}\, A_{\substack{45\\45}}\right) = A^2\, a_{11}^3\, A_{\substack{45\\45}} \text{ or } A^3\, A^2_{\substack{2345\\2345}}\, A_{\substack{45\\45}}.
$$

The product

$$
\begin{vmatrix}
A_{\substack{12\\12}} & A_{\substack{12\\13}} & A_{\substack{12\\14}} & A_{\substack{12\\23}} & A_{\substack{12\\24}} & A_{\substack{12\\34}} \\
A_{\substack{13\\12}} & A_{\substack{13\\13}} & A_{\substack{13\\14}} & A_{\substack{13\\23}} & A_{\substack{13\\24}} & A_{\substack{13\\34}} \\
A_{\substack{14\\12}} & A_{\substack{14\\13}} & A_{\substack{14\\14}} & A_{\substack{14\\23}} & A_{\substack{14\\24}} & A_{\substack{14\\34}} \\
A_{\substack{15\\12}} & A_{\substack{15\\13}} & A_{\substack{15\\14}} & A_{\substack{15\\23}} & A_{\substack{15\\24}} & A_{\substack{15\\34}} \\
A_{\substack{23\\12}} & A_{\substack{23\\13}} & A_{\substack{23\\14}} & A_{\substack{23\\23}} & A_{\substack{23\\24}} & A_{\substack{23\\34}} \\
0 & 0 & 0 & 0 & 0 & 1
\end{vmatrix}
\cdot
\begin{vmatrix}
A_{\substack{34\\34}} & -A_{\substack{34\\24}} & A_{\substack{34\\23}} & A_{\substack{34\\14}} & -A_{\substack{34\\13}} & A_{\substack{34\\12}} \\
-A_{\substack{24\\34}} & A_{\substack{24\\24}} & -A_{\substack{24\\23}} & -A_{\substack{24\\14}} & A_{\substack{24\\13}} & -A_{\substack{24\\12}} \\
A_{\substack{23\\34}} & -A_{\substack{23\\24}} & A_{\substack{23\\23}} & A_{\substack{23\\14}} & -A_{\substack{23\\13}} & A_{\substack{23\\12}} \\
A_{\substack{14\\34}} & -A_{\substack{14\\24}} & A_{\substack{14\\23}} & A_{\substack{14\\14}} & -A_{\substack{14\\13}} & A_{\substack{14\\12}} \\
-A_{\substack{13\\34}} & A_{\substack{13\\24}} & -A_{\substack{13\\23}} & -A_{\substack{13\\14}} & A_{\substack{13\\13}} & -A_{\substack{13\\12}} \\
A_{\substack{12\\34}} & -A_{\substack{12\\24}} & A_{\substack{12\\23}} & A_{\substack{12\\14}} & -A_{\substack{12\\13}} & A_{\substack{12\\12}}
\end{vmatrix}
$$

$$
= \begin{vmatrix}
A.a_{55} & 0 & 0 & 0 & 0 & 0 \\
0 & A.a_{55} & 0 & 0 & 0 & 0 \\
0 & 0 & A.a_{55} & 0 & 0 & 0 \\
A.a_{25} & A.a_{35} & A.a_{45} & 0 & 0 & 0 \\
0 & 0 & 0 & A.a_{55} & 0 & 0 \\
A_{\substack{34\\12}} & -A_{\substack{24\\12}} & A_{\substack{23\\12}} & A_{\substack{14\\12}} & -A_{\substack{13\\12}} & A_{\substack{12\\12}}
\end{vmatrix} = 0,
$$

$$
\therefore \left(A_{\substack{12\\12}}\, A_{\substack{13\\13}}\, A_{\substack{14\\14}}\, A_{\substack{15\\23}}\, A_{\substack{23\\24}}\right) = 0.
$$

There are $\{6.5\}^2 = \{30\}^2$ vanishing minors of order five.

SYRACUSE UNIVERSITY, *Sept.* 1, 1897.

A Theorem in Determinants.

BY W. H. METZLER, PH. D.

The theorem in question is an extension of one given by Binet in Journn. de l'Ec. Polyt. IX, cah. 16, pp. 280–302.*

If we have the four sets of n quantities

$$\begin{array}{llll} a_{11} & a_{12} & a_{13} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots\ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} \ldots\ldots & a_{3n} \\ a_{41} & a_{42} & a_{43} \ldots\ldots & a_{4n}, \end{array}$$

and $\sum A_{\substack{1234\\1234}}$† denotes the sum of the n_4 determinants of the fourth order obtainable from them, $\sum A_{\substack{123\\123}}$ denotes the like sum obtainable from the first three sets, etc. Then Binet proved that

$$\sum a_{11}\sum A_{\substack{23\\23}} + \sum a_{21}\sum A_{\substack{13\\23}} + \sum a_{31}\sum A_{\substack{12\\23}} = \sum A_{\substack{123\\123}}$$

and

$$\sum a_{11}\sum A_{\substack{234\\234}} + \sum a_{21}\sum A_{\substack{134\\234}} + \sum a_{31}\sum A_{\substack{124\\234}} + \sum a_{41}\sum A_{\substack{123\\234}} = 0.$$

These may be extended to the case where instead of four we have m sets of n quantities, and instead of taking the sets one and $m - 1$ at a time, we take them l and $m - l$ at a time.

The proof of the general case will be better understood if we first consider a special case.

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* Vide Muir's "The Theory of Determinants in the Historical Order of its Development," Part I, pp. 86–91.

† It is to be observed that the notation used throughout this paper is inclusive, i. e. $A_{\substack{rst\ldots\\uvw\ldots}}$ denotes the minor formed by the constituents in the intersection of the $r, s, t \ldots\ldots$ rows with the $u, v, w \ldots\ldots$ columns.

If

$$A=\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{vmatrix}$$

then

$$\begin{aligned}
&\left(A_{\substack{12\\12}}+A_{\substack{12\\13}}+\dots+A_{\substack{12\\56}}\right)\left(A_{\substack{34\\12}}+A_{\substack{34\\13}}+\dots+A_{\substack{34\\56}}\right)\\
-&\left(A_{\substack{13\\12}}+A_{\substack{13\\13}}+\dots+A_{\substack{13\\56}}\right)\left(A_{\substack{24\\12}}+A_{\substack{24\\13}}+\dots+A_{\substack{24\\56}}\right)\\
+&\left(A_{\substack{14\\12}}+A_{\substack{14\\13}}+\dots+A_{\substack{14\\56}}\right)\left(A_{\substack{23\\12}}+A_{\substack{23\\13}}+\dots+A_{\substack{23\\56}}\right)\\
+&\left(A_{\substack{23\\12}}+A_{\substack{23\\13}}+\dots+A_{\substack{23\\56}}\right)\left(A_{\substack{14\\12}}+A_{\substack{14\\13}}+\dots+A_{\substack{14\\56}}\right)\\
-&\left(A_{\substack{24\\12}}+A_{\substack{24\\13}}+\dots+A_{\substack{24\\56}}\right)\left(A_{\substack{13\\12}}+A_{\substack{13\\13}}+\dots+A_{\substack{13\\56}}\right)\\
+&\left(A_{\substack{34\\12}}+A_{\substack{34\\13}}+\dots+A_{\substack{34\\56}}\right)\left(A_{\substack{12\\12}}+A_{\substack{12\\13}}+\dots+A_{\substack{12\\56}}\right)\\
=&\,2\Big(A_{\substack{1234\\1234}}+A_{\substack{1234\\1235}}+A_{\substack{1234\\1236}}+A_{\substack{1234\\1245}}+A_{\substack{1234\\1246}}\\
+&\quad A_{\substack{1234\\1256}}+A_{\substack{1234\\1345}}+A_{\substack{1234\\1346}}+A_{\substack{1234\\1356}}+A_{\substack{1234\\1456}}\\
+&\quad A_{\substack{1234\\2345}}+A_{\substack{1234\\2346}}+A_{\substack{1234\\2356}}+A_{\substack{1234\\2456}}+A_{\substack{1234\\3456}}\Big).
\end{aligned}$$

For if we expand the left-hand member, the sum of corresponding terms in the products vanishes or not according as the factors of the terms have or have not a column in common, i. e. according as there is or is not a number common to the lower lines of the suffixes of both factors.

Thus

$$\begin{aligned}
&A_{\substack{12\\12}}A_{\substack{34\\12}}-A_{\substack{13\\12}}A_{\substack{24\\12}}+A_{\substack{14\\12}}A_{\substack{23\\12}}+A_{\substack{23\\12}}A_{\substack{14\\12}}-A_{\substack{24\\12}}A_{\substack{13\\12}}+A_{\substack{34\\12}}A_{\substack{12\\12}}=0,\\
&A_{\substack{12\\13}}A_{\substack{34\\13}}-A_{\substack{13\\13}}A_{\substack{24\\13}}+A_{\substack{14\\13}}A_{\substack{23\\13}}+A_{\substack{23\\13}}A_{\substack{14\\13}}-A_{\substack{24\\13}}A_{\substack{13\\13}}+A_{\substack{34\\13}}A_{\substack{12\\13}}=0,
\end{aligned}$$

etc.

$$\begin{aligned}
&A_{\substack{12\\12}}A_{\substack{34\\34}}-A_{\substack{13\\12}}A_{\substack{24\\34}}+A_{\substack{14\\12}}A_{\substack{23\\34}}+A_{\substack{23\\12}}A_{\substack{14\\34}}-A_{\substack{24\\12}}A_{\substack{13\\34}}+A_{\substack{34\\12}}A_{\substack{12\\34}}=A_{\substack{1234\\1234}},\\
&A_{\substack{12\\13}}A_{\substack{34\\24}}-A_{\substack{13\\13}}A_{\substack{24\\24}}+A_{\substack{14\\13}}A_{\substack{23\\24}}+A_{\substack{23\\13}}A_{\substack{14\\24}}-A_{\substack{24\\13}}A_{\substack{13\\24}}+A_{\substack{34\\13}}A_{\substack{12\\24}}=-A_{\substack{1234\\1234}},
\end{aligned}$$

etc.

It is easily seen that $A_{\substack{1234\\1234}}$ as well as each of the other terms on the right

will be obtained in $\frac{4 \cdot 3}{1 \cdot 2} = 6$ different ways, four of which will give a positive and two a negative sign, since in the set of combinations 1234, 1324, 1423, 2314, 2413, 3412 there are four having an even number of inversions and two having an odd number of inversions. Hence the truth of the equation.

In general if

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix},$$

then

$$(-1)^{\nu} \sum_{\delta=1}^{\delta=n_l} A_{\substack{(n|\underset{\alpha}{m}|\underset{1}{l}) \\ (n|\underset{\delta}{l})}} \cdot \sum_{\delta=1}^{\delta=n_{m-l}} A_{\substack{(n|\underset{\alpha}{\overline{m}}|\underset{1}{l}) \\ (n|\underset{\delta}{m-l})}}$$

$$+ (-1)^{\nu} \sum_{\delta=1}^{\delta=n_l} A_{\substack{(n|\underset{\alpha}{m}|\underset{2}{l}) \\ (n|\underset{\delta}{l})}} \cdot \sum_{\delta=1}^{\delta=n_{m-l}} A_{\substack{(n|\underset{\alpha}{\overline{m}}|\underset{2}{l}) \\ (n|\underset{\delta}{m-l})}},$$

$$+ \text{ etc.}$$

$$= \phi(m, l) \cdot \sum_{\beta=1}^{\beta=n_m} A^{*}_{\substack{(n|m) \\ (n|\underset{\beta}{\overset{\alpha}{m}})}},$$

or

$$\sum_{\gamma=1}^{\gamma=m_l} (-1)^{\nu} \left[\sum_{\delta=1}^{\delta=n} A_{\substack{(n|\underset{\alpha}{m}|\underset{\gamma}{l}) \\ (n|\underset{\delta}{l})}} \sum_{\delta=1}^{\delta=n_{m-l}} A_{\substack{(n|\underset{\alpha}{\overline{m}}|\underset{\gamma}{l}) \\ (n|\underset{\delta}{m-l})}} \right]$$

$$= \phi(m, l) \sum_{\beta=1}^{\beta=n_m} A_{\substack{(n|m) \\ (n|\underset{\beta}{\overset{\alpha}{m}})}},$$

where ν denotes the number of inversions in $(n|\underset{\alpha}{m}|\underset{\gamma}{l})(n|\underset{\alpha}{\overline{m}}|\underset{\gamma}{l})$,* and $\phi(m, l)$ denotes the excess of the number of combinations in the set

$$(n|\underset{\beta}{m}|\underset{1}{l})(n|\underset{\beta}{\overline{m}}|\underset{1}{l}),\ (n|\underset{\beta}{m}|\underset{2}{l})(n|\underset{\beta}{\overline{m}}|\underset{2}{l}),\ \dots\ (n|\underset{\beta}{m}|\underset{\delta}{l})(n|\underset{\beta}{\overline{m}}|\underset{\delta}{l}),\quad \delta = n_l$$

which have an even number of inversions over those which have an odd number.*

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* For an explanation of this notation vide this Journal, vol. XX, No. 3.

For if we expand the left-hand member as before, it is easily seen that the sum of corresponding terms of the products will vanish or not according as $(n\,|\,\underset{\delta}{l})$ and $(n\,|\,m \underset{\delta}{-} l)$ have or have not any numbers in common. Any term $A_{\substack{(n\,|\,m) \\ (n\,|\,\overset{\alpha}{\underset{\beta}{m}})}}$ which does not vanish will be obtained in m_l ways from those particular sums of corresponding terms of products which are given by

$$\sum_{\gamma=1}^{\gamma=m_l}(-1)^{\gamma}\left[A_{\substack{(n\,|\,m\,|\,l) \\ (n\,|\,\overset{\alpha}{\underset{\beta}{m}}\,|\,\overset{\gamma}{\underset{\delta}{l}})}} \cdot A_{\substack{(n\,|\,m\,|\,l) \\ (n\,|\,\overset{\alpha}{\underset{\beta}{m}}\,|\,\overset{\gamma}{\underset{\delta}{l}})}}\right], \ (\delta = 1, 2, \ldots, m_l).$$

The coefficient of $A_{\substack{(n\,|\,m) \\ (n\,|\,\overset{\alpha}{\underset{\beta}{m}})}}$ is therefore $\phi(m, l)$.*

The theorem is thus established.

SYRACUSE UNIVERSITY, *February* 10, 1898.

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* Vide author's paper read before the Am. Math. Soc., Feb. 26, 1898, and to appear in a later number of this Journal.

On the Perfect Groups.

By G. A. Miller.

The necessary and sufficient condition that a group is *solvable* is that its α^{th} derivative (derived group) is unity.* When no lower than the α^{th} derivative is unity, the $(\alpha-1)^{\text{th}}$ derivative must be an Abelian group whose order exceeds unity. All the derivatives which are lower than the $(\alpha-1)^{\text{th}}$ must then be non-Abelian. Dedekind calls the first derivative the commutator subgroup. The more general notation which we employ is due to Lie.

When a group is *insolvable* its α^{th} derivative must be a *perfect*† group whose order exceeds unity. The factors of composition of this perfect group include all the composite factors of composition of the original insolvable group. Hence the study of insolvable groups is reduced to that of perfect groups. In other words, an insolvable group is either perfect or it contains one and only one perfect group as a characteristic subgroup with respect to which its quotient group is solvable.

Since a perfect group is identical with its derivatives, it cannot be isomorphic to any Abelian group whose order exceeds unity. Conversely, if a group is not isomorphic to any Abelian group whose order exceeds unity, it must coincide with its first derivate, and hence it must be perfect. The totality of perfect groups, therefore, includes that of simple groups of a composite order, but it is included in that of insolvable groups.

It is easy to see that no two of these three totalities are identical, for the direct product of any number of simple groups of composite orders is evidently a compound perfect group, while the direct product of a perfect and a solvable group is insolvable without being perfect. All the symmetric groups whose

*Quarterly Journal of Mathematics, vol. 28, p. 268; cf. Frobenius, Sitzungsberichte der Berliner Akademie, 1896, p. 1848.

†A perfect group is identical with its first derivative (Lie).

orders exceed 24 are also insolvable without being perfect. We proceed to consider some general properties of any perfect group.

Theorem I.—Every perfect group has an α, 1 *isomorphism to a simple group of composite order.*

When $\alpha = 1$ the perfect group is simple and of a composite order, and vice versa. When the perfect group is compound it must have an i, 1 isomorphism to a *perfect* group of lower order, for if this group of lower order were imperfect the original group would have to be imperfect. We repeat this process if the given perfect group of lower order is not simple. We shall thus finally arrive at a simple group of composite order to which the original perfect group has an α, 1 isomorphism. The order of this simple group is a factor of composition of the original group.

Corollary I.—If an imprimitive substitution group is perfect it must permute all its systems of imprimitivity according to a perfect group.

Corollary II.—If a perfect group has prime factors of composition it must contain a solvable characteristic subgroup whose order is the product of all these prime factors.

Theorem II.—If a perfect group is represented as an intransitive substitution group, all its transitive constituents must be perfect.

If a transitive constituent were imperfect the first derivative of the group could not include all the substitutions of this constituent. It could therefore not contain all the substitutions of the group which is supposed to be perfect. This is impossible.

Corollary.—A transitive constituent of a perfect substitution group contains only positive substitutions.

Theorem III.—If a transitive substitution group of a prime degree is perfect it is also simple.

If such a group were compound it would contain a transitive invariant (self-conjugate) subgroup whose order would be less than the order of the group. This subgroup would contain all the cyclical substitutions of the prime degree (p) that are contained in the entire group. With respect to this subgroup the entire group would have an α, 1 isomorphism to a cyclical group of order $(p-1) \div \beta$, β being an integer. As a perfect group cannot have such an isomorphism, the given group cannot be compound.

Corollary.—A transitive group of a prime degree cannot contain more than one composite factor of composition.

If it contained two such factors both of them would be factors of composition of its perfect characteristic subgroup with respect to which it is isomorphic to a solvable group. This is impossible, since this characteristic subgroup is transitive.

Theorem IV.—If a transitive substitution group of degree $2p$, p being any prime number, is perfect, it is either simple or imprimitive. In the latter case it must contain p systems of imprimitivity, and it must permute them according to a simple group of degree p.

Since all the groups whose degrees are less than five are solvable, we may assume that p is odd. If the required perfect group were primitive and compound, all its substitutions of order p would be contained in an invariant transitive subgroup whose order would be less than the order of the group.* With respect to this subgroup the perfect group would have to be isomorphic to an Abelian group. As this is impossible, the given perfect group is simple if it is primitive. If the required perfect group is imprimitive it must permute its systems of imprimitivity according to a perfect transitive group of degree p. We proved above that such a group is simple.

Theorem V.—Any simple quotient group of a compound substitution group of degree n may be represented as a transitive group whose degree is less than n.

If the given group (G) of degree n is intransitive, it must contain at least one transitive constituent which has an α, 1 (α being an integer) isomorphism to the given quotient group, since the latter is simple. Hence we may confine our attention to the case when G is transitive, and we may suppose that n is the smallest number of elements by means of which G can be represented transitively.

If a subgroup of G, which contains all its substitutions that do not contain a given element, corresponds to only a part of the operators of the simple quotient group (S), it is possible to represent S as a transitive group of degree $n \div m$, $m > 1$. If this subgroup is simply isomorphic to S, it is possible to represent S as a transitive group whose degree cannot exceed $n - 1$. If this subgroup is multiply isomorphic to S, we may use it or one of its transitive constituents in place of G. In either case the new transitive group of which S is also a quotient group is of lower order and of lower degree than G. As this process cannot be repeated indefinitely the theorem is proved.

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* Jordan, Bulletin de la Société Mathématique de France, t. 1, p. 41.

Corollary.—If a quotient group of G is perfect and compound, G is isomorphic to some simple group of composite order which can be represented by a smaller number of elements than is required to represent G.

By means of these theorems we may readily determine a large number of perfect groups. As all the simple groups of composite order are perfect we need not consider these. We proceed to seek all the compound perfect groups which may be represented as substitution groups with 11 or a smaller number of elements. According to the preceding theorems the lowest possible degree of such a group is 8, and if a compound group of this degree is perfect it must be primitive. As 8 is equal to a prime number plus 3, the order of a primitive group of this degree is not divisible by 5 unless it is either the alternating or the symmetric group.* The composite factor of composition of the required group must therefore be 168 and its order cannot be less than $168 \times 8 = 1344$, since it must contain a transitive invariant subgroup. There is only one primitive group of this degree that satisfies the last condition and does not contain the alternating group, viz. the well-known triply transitive group of order 1344.† That this group is perfect follows directly from the facts that it contains no invariant subgroup of order 168, and the seven operators of order 2 in its invariant subgroup of order 8 are conjugate. Hence *there is one and only one compound perfect group of degree* 8, *and there is no such group for any lower degree.*

If a group of degree 9 were compound and perfect it would also be primitive according to the given theorems. As its order could not be divisible by 5, and as it could not contain two composite factors of composition, it would have to be isomorphic to the simple group of order 168 with respect to a transitive invariant subgroup. It could not be isomorphic to the simple group of order 504, since this cannot be represented by less than 9 elements. Hence the order of the required group could not be less than $168.9 = 1512$. But the primitive group of this order does not contain an invariant subgroup of order 9, and there is no larger primitive group of degree 9 that does not include the alternating group of this degree. Hence *there is no compound perfect group of degree* 9.

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* Jordan, loc. cit.

† This group is given by Kirkman, Proceedings of the Literary and Philosophical Society of Manchester (1863), vol. 3; by Jordan, Comptes Rendus, vol. 73; by Noether, Mathematische Annalen, vol. 15; and by others. It is singular that Wiman supposed that he established its existence for the first time, Nachrichten, Göttingen (1897), p. 58.

Since a perfect group of a prime degree must be simple, and there is only one simple group of degree 5 and all the groups of a lower degree are solvable, there can be only one compound perfect intransitive group of degree 10. It is of order 3600, being the direct product of two simple groups of order 60. An imprimitive compound perfect group of this degree must contain 5 systems of imprimitivity, and it must permute them according to the alternating group of degree 5. Its head can contain only positive substitutions and its order (2^a) must satisfy the congruence

$$2^a \equiv 1 \bmod. 5 \qquad 1 < a < 5.$$

Hence $a = 4$, and the order of the imprimitive group is $16.60 = 960$. There is only one imprimitive group of this degree and order.* If it were not perfect it would contain an invariant subgroup of order 60, since no subgroup, whose order exceeds unity, that is contained in the head can be invariant. Each substitution of an invariant subgroup of order 60 would have to be commutative with every substitution of the head. As this condition could not be satisfied by the substitution of order 5, the given imprimitive group must be a compound perfect group.

If a primitive group of degree 10 were compound and perfect it would be isomorphic to a simple group of a lower degree whose order is not divisible by 7, since a cyclical substitution of a prime degree (p) cannot occur in any primitive groups except those of degrees p, $p+1$, $p+2$, and those of higher degrees which include the alternating group of their own degree. As a transitive group would correspond to identity in the isomorphic simple group, and the order of such a simple group would be divisible by 5, the order of the required primitive group would be divisible by 25. This is clearly impossible, since a group of degree 10 and order 25 must contain a cyclical substitution of order 5. Hence *there are two and only two compound perfect groups of degree* 10, *the one is intransitive and the other is imprimitive.*

The compound perfect groups of degree 11 must be intransitive and their transitive constituents must be simple and of degrees 5 and 6. Hence there are two such groups, viz. the direct product of the alternating groups of these degrees and the direct product of the alternating group of degree 5 and the primitive group of degree 6 which is simply isomorphic to it. Their orders are 21600 and

* Cole, Quarterly Journal of Mathematics, vol. 27, p. 42.

3600 respectively. We give below the enumerations of the simple, the perfect, and the insolvable groups which may be represented by 11 or a smaller number of elements. The lowest order of a compound perfect group is 120. There is only one such group of this order. As a substitution group it can be represented by 24, but by no smaller number of elements.

Degree	5	6	7	8	9	10	11
Number of simple groups*	2	2	3	2	2	4	5
Number of perfect groups	1	2	2	3	2	6	6
Number of insolvable groups	2	4	6	16	41	106	228

CHICAGO, *Dec.* 1897.

* This enumeration includes all the possible substitution groups of the given degrees. If we regard these groups as operation groups they are not all distinct.

On Darboux Lines on Surfaces.

By James G. Hardy.

1. The problem which forms the subject of this note was first proposed and discussed by M. Darboux in an article entitled "Des courbes tracées sur une surface, et dont la sphère osculatrice est tangente en chaque point à la surface," which appeared in the Comptes Rendus for 1871, and in which he deduced the differential equation of these curves, which is of the second order, and integrated it in the cases of the quadric surfaces and cyclides. Later in the same year Enneper published in the Göttinger Nachrichten a paper in which the geometric signification of the various terms in the differential equation of these lines was pointed out, and in which he deduced their characteristic property that at any point of the curve the radius of its osculating sphere is equal to the radius of curvature of the normal section of the surface having the same tangent. He also showed that if one of these lines be a line of curvature of the surface on which it lies, the surface is the envelope of a sphere of variable radius whose centre describes an arbitrary skew curve. In 1875 an article by M. Ribacour appeared in the Comptes Rendus, in which the author showed that if we trace on a surface S a curve of the kind considered, then each of the osculating planes of this curve cuts S along a section superosculated by a circle. The last paper to appear on this subject is due to M. Cosserat, and is found in the Comptes Rendus for 1895. In it he applies to the investigation of these lines methods analogous to those used in the investigation of geodesics, employing the theory of integrals of determinate form treated for geodesics in Livre VII of the "Leçons" of M. Darboux. I have called these lines Darboux lines because first defined and treated by M. Darboux; M. Cosserat named them D-lines, evidently for the same reasons.

2. Suppose that at any point P of a Darboux line D on a surface we construct a trihedron T_0 having its vertex at P and the normal to the surface as its

axis of z; let ω be the angle which the tangent to D makes with the axis of x of T_0, and let ρ and R be the radius of the osculating sphere and the radius of curvature of D at P respectively. Since by definition the osculating sphere of D is tangent to the surface, its center will be at the intersection of the normal to the surface and the polar line at P. Then, by considering the right triangle whose vertices are the point P, the center of curvature K, and the center of the osculating sphere, $M(x_0 y_0 z_0)$, and applying Meusnier's theorem, we have: the radius of curvature R_n of the normal section of the surface having the same tangent as D, is equal to the radius of the osculating sphere of D at the point considered. Enneper, Gött. Nach., 1871.

Denote by T the radius of torsion of D, by ϕ the angle between the principal normal to D and the normal to the surface at P, and use the formulæ given by Darboux (Théorie générale, II, 359) for the coordinates x_0, y_0, z_0 of the center of the osculating sphere of any line traced on a surface. We have

$$z_0 = R\cos\phi - T\frac{dR}{ds}\sin\phi = 0,$$

from which

$$\tan\phi = -T\frac{1}{R}\,\frac{dR}{ds} \quad \text{and} \quad T = -\frac{\tan\phi}{\dfrac{d}{ds}\log R}.$$

If in the last equation we make $R = \text{const.}$, we will have $T = \infty$ unless $\phi = 0$. Hence *when the radius of curvature of a Darboux line is constant, the line is either plane or geodesic.*

Conversely, when $\phi = 0$ we must have $R = \text{const.}$, since $T \neq 0$; that is, *when a Darboux line is geodesic its radius of curvature is constant.*

If we use the formula

$$\text{geodesic torsion} \equiv \frac{1}{T_g} = \left[\frac{1}{R_1} - \frac{1}{R_2}\right]\sin\omega\cos\omega = \frac{1}{T} - \frac{d\phi}{ds},$$

we find for the geodesic torsion of D

$$\frac{1}{T_g} = -\frac{1}{\tan\phi}\,\frac{d}{ds}\log R_n.$$

Then for $\phi \neq 0$, $\dfrac{1}{T_g}$ can only be zero when R_n is constant. But when $\dfrac{1}{T_g} = 0$ the line D is a line of curvature; hence if a line of curvature be a Darboux line, the principal radius of curvature corresponding will be constant. Now in this

case one sheaf of the evolute of the surface reduces to a line and, as Monge proved, the surface may be determined as the envelope of a sphere of variable radius whose center describes an arbitrary skew curve. (Enneper, Gött. Nach. l. c.) This same result can be obtained in a different way, as will be seen later. If both systems of lines of curvature be Darboux lines the surface will be a cyclide of Dupin; for M. Bonnet has demonstrated (Journal de l'Ecole Polytechnique, XLII) that if a surface be such that along each line of curvature the principal radius corresponding to that line be constant, the surface is a cyclide of Dupin.

In a note published in the Bulletin des Sciences Mathématiques, 21, 1898, M. Demartres gives some formulæ concerning skew curves which have interesting forms when applied to Darboux lines. The distance between the centers of the osculating spheres at the extremities of an arc SS' of such a line is

$$\delta = \frac{R_n}{T}\,\frac{dR_n}{dR}\,ds;$$

and the angle ψ under which these spheres cut is

$$\psi = \frac{ds}{T}\,\frac{d\log R_n}{d\log R}.$$

Along a line of curvature we have $\psi = 0$, but (unless the Darboux line be plane) this requires that R_n be constant, or that $\delta = 0$, in case $R \neq$ constant. That is, if a line of curvature be a Darboux line it is a spherical line or a line of constant curvature.

Using the definitions of absolute and relative spherical torsion given by M. Demartres, we may define Darboux lines as those lines whose absolute spherical torsion is equal to their relative spherical torsion.

The formula just used for geodesic torsion leads to a neat form in the case of Darboux lines on an ellipsoid referred to its lines of curvature. Employing the values given later for $\sin\omega$, $\cos\omega$, ds^2 and $\left(\frac{du}{dv}\right)^2$, we get

$$\frac{1}{T_g} = c\,.\,(u-v)\sqrt{\frac{(u-\varkappa)(v-\varkappa)}{U\quad V}}. \qquad c = \text{a constant}$$

This puts into evidence the property of lines of curvature made use of.

The line joining the points P and M generates a ruled surface Σ. The locus of the centers of the osculating spheres of D is the edge of regression of the polar

surface, and its osculating plane, which is the normal plane of D, remains tangent to Σ. Therefore the locus of the centers of the osculating spheres of D is an asymptotic line on Σ. Then along any Darboux line the normals to the surface generate a ruled surface on which we know an asymptotic line.

3. The general differential equation of Darboux lines, as given by M. Darboux, is

$$(1) \qquad \frac{3d^2s}{ds} = \frac{2\Sigma d^2x \,.\, d\dfrac{\partial F}{\partial x} + \Sigma dx \,.\, d^2\dfrac{\partial F}{\partial x}}{\Sigma\, dx \,.\, d\dfrac{\partial F}{\partial x}},$$

where $F=0$ is the equation of the surface on which the lines lie, and the sign Σ indicates summation with regard to the three coordinates x, y, z. This equation in the case of surfaces of the 2nd degree takes the form

$$(2) \qquad \varkappa \,.\, ds^2 = \Sigma\left[dx \,.\, d\frac{\partial F}{\partial x}\right], \qquad \varkappa = \text{a constant.}$$

or, introducing elliptic coordinates,

$$(3) \qquad \frac{(u-\varkappa)\,du^2}{(a^2-u)(b^2-u)(c^2-u)} = \frac{(v-\varkappa)\,dv^2}{(a^2-v)(b^2-v)(c^2-v)}.$$

If we denote by R_n the radius of curvature of a normal section of an ellipsoid, and if the lines of curvature are taken as coordinate lines, R_n will be given by

$$(4) \qquad R_n = \frac{E\lambda^2+G}{D\lambda^2+D''}. \qquad \lambda = \frac{du}{dv}.$$

Let the plane of the normal section considered be passed through the tangent to a Darboux line; calculate λ^2 from (3), and remember that (Darboux, Théorie Générale, II, 379)

$$D = -\frac{\sqrt{abc}\,(u-v)^2}{f(u)\sqrt{-f(u)f(v)}}; \quad D'' = \frac{\sqrt{abc}\,(u-v)^2}{f(v)\sqrt{-f(u)f(v)}};$$

then

$$(5) \qquad R_n = -\varkappa\frac{\sqrt[4]{R_1R_2}}{\sqrt[4]{abc}},$$

where R_1 and R_2 are the principal radii of curvature at the point considered. We have here a result stated by Enneper: The radius of curvature of the normal section of an ellipsoid having the same tangent as a Darboux line is

proportional to the 4th root of the product of the principal radii of curvature at the point considered.

From this result follows another still more interesting. If we calculate the distance from the origin to the tangent plane at any point of an ellipsoid, we find

$$\frac{1}{\delta} = \frac{\sqrt[4]{R_1 R_2}}{\sqrt[4]{abc}}. \tag{6}$$

Taking account of (5) and (6), and remembering that for Darboux lines the radius of curvature of the normal section through the tangent is equal to the radius of the osculating sphere, we have

$$\rho = -\varkappa \frac{1}{\delta};$$

that is, the radius ρ of the osculating sphere is inversely proportional to the distance from the center of the ellipsoid to the tangent plane. A better way of stating this same fact is: *the product of the radius of the osculating sphere and the distance from the origin to the tangent plane is constant.*

$$\rho\delta = \text{const.} \tag{7}$$

If the Darboux line be a polhodie, $\delta =$ constant, and hence $\rho =$ constant. The line is therefore spherical, and also (Théorie Générale, II, 381) each normal section tangent to the Darboux line at one of its points is superosculated at that point by a circle.

For the paraboloids $F = \frac{2Xx}{A} + \frac{2Yy}{B} - Z - z = 0$, we find the distance δ to be

$$\delta = \frac{-z}{1 + \frac{4x^2}{A^2} + \frac{4y^2}{B^2}}.$$

Equation (7) then takes the form

$$\rho\delta = \text{constant} \, . \, z.$$

4. Equation (2) has a very interesting geometric signification, communicated to me for the case of the ellipsoid by Mr. Pell, and which I believe is new. Suppose the surface of the 2nd degree considered be of the form

$$F(x, y, z) = \frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} - 1 = 0,$$

then (2) gives

$$\varkappa = \left[\left(\frac{dx}{ds}\right)^2 \frac{1}{a^2} + \left(\frac{dy}{ds}\right)^2 \frac{1}{b^2} + \left(\frac{dz}{ds}\right)^2 \frac{1}{c^2}\right],$$

which, since $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ are the direction cosines of the tangent at (x, y, z), may be written

$$(8) \qquad \varkappa = \left[\frac{\cos^2\alpha}{a^2} + \frac{\cos^2\beta}{b^2} + \frac{\cos^2\gamma}{c^2}\right].$$

If we denote by d the semidiameter whose direction cosines are $\cos\alpha$, $\cos\beta$, $\cos\gamma$, the coordinates of its end points are $d.\cos\alpha$, $d.\cos\beta$, $d.\cos\gamma$; and, since these coordinates satisfy the equation of the surface,

$$(9) \qquad d^2\left[\frac{\cos^2\alpha}{a^2} + \frac{\cos^2\beta}{b^2} + \frac{\cos^2\gamma}{c^2}\right] = 1.$$

(8) and (9) combined give

$$d = \text{constant}.$$

That is, *if at any point of a Darboux line of $F = 0$ a tangent be drawn, and if we construct the semidiameter parallel to this tangent, the length of this semidiameter is constant along the considered Darboux line.*

5. The different Darboux lines on surfaces of the 2nd degree are obtained by giving different values to the constant $\varkappa$ which enters (3). Suppose in this equation we make $\varkappa = 0$, then

$$\frac{u\,du^2}{U} - \frac{v\,dv^2}{V} = 0,$$

where $U = (a^2 - u)(b^2 - u)(c^2 - u)$ and $V = (a^2 - v)(b^2 - v)(c^2 - v)$. But we have

$$ds^2 = \frac{u - v}{4}\left[\frac{u\,du^2}{U} - \frac{v\,dv^2}{V}\right];$$

hence

$$ds^2 = 0.$$

That is, when $\varkappa = 0$ the Darboux lines are lines of length zero on the surface.

If in (3), after extracting the square root, we put $\varkappa = \infty$, we find

$$(10) \qquad \frac{du}{\sqrt{U}} = \frac{dv}{\sqrt{V}},$$

which is the equation of the right lines on the surface. This may be verified as follows: the equations of the right lines on a quadric are

$$\frac{x}{a} = \frac{z}{c}\, i \,.\cos\phi \pm \sin\phi, \quad \frac{y}{b} = \frac{z}{c}\, i\,.\sin\phi \pm \cos\phi.$$

From these

$$\frac{dx^2}{a^2} + \frac{dy^2}{b^2} + \frac{dz^2}{c^2} = 0,$$

which, on substituting for dx, dy, dz their values in terms of du and dv, becomes (10).

6. Consider now an ellipsoid with three unequal axes $a > b > c$, and let us determine for what values of $\varkappa$ the Darboux lines will be real. The elliptic coordinates which enter our equations symmetrically may be distinguished by taking

$$c^2 < u < b^2; \qquad b^2 < v < a^2.$$

Under these conditions $\frac{1}{U}$ is negative and $\frac{1}{V}$ is positive, and consequently in order that the differential equation

$$du\sqrt{\frac{u-\varkappa}{U}} = dv\sqrt{\frac{v-\varkappa}{V}}$$

of the Darboux lines be real, we must have

$$u - \varkappa < 0; \qquad v - \varkappa > 0.$$

That is, the constant $\varkappa$ must satisfy the inequalities

$$c^2 < \varkappa < a^2. \tag{11}$$

That the lines of length zero and the right lines on an ellipsoid are imaginary would appear from (11) if it were not otherwise obvious.

According as the constant $\varkappa$ has different values the following cases may arise:

1. $b^2 > \varkappa > c^2$,
2. $b^2 \leqq \varkappa < a^2$,
3. $\varkappa^2 = a^2$, or b^2, or c^2.

Now if ω be the angle made by the tangent to the considered Darboux line and the axis of x of a trihedron constructed as in §1, we have

$$\cos\omega = \sqrt{E}\,\frac{du}{ds}, \quad \sin\omega = \sqrt{G}\,\frac{dv}{ds}, \quad \tan\omega = \sqrt{\frac{G}{E}}\,\frac{dv}{du};$$

determining $\frac{dv}{du}$ from (3), we find

$$\tan\omega = -\frac{v}{u}\sqrt{\frac{U(u-\varkappa)}{V(v-\varkappa)}}. \tag{12}$$

CASE (1). $\varkappa$ has a u-value $\varkappa = u$, which, substituted in (12), gives

$$\tan\omega = 0, \quad \omega = 0.$$

That is, the Darboux line $\varkappa = u$ is perpendicular to the line of curvature $u = \varkappa$.

CASE (2). When $\varkappa$ has a v-value $\varkappa = v$, equation (12) gives

$$\tan\omega = \infty, \quad \omega = 90°.$$

That is, the Darboux line $\varkappa = v$ is perpendicular to the line of curvature $v = \varkappa$.

CASE (3). To fix the ideas, take $\varkappa = b^2$. The differential equation of the Darboux lines becomes

$$(a^2 - v)(c^2 - v)\,du^2 = (a^2 - u)(c^2 - u)\,dv^2,$$

which is the differential equation of the real circular sections on the ellipsoid. For $\varkappa^2 = a^2$ or c^2 we get the other two systems of circular sections, imaginary of course. (Darboux, C. R., 1871.)

7. In the differential equation of the Darboux lines if we consider u and v as the coordinates of a given point M of the curve, the equation for a given value of $\varkappa$ determines the direction of the curve at M. Conversely, if the direction be given, the equation determines the value of the parameter $\varkappa$. Write

$$ds^2 = \frac{u-v}{4}\left[\frac{u\,du^2}{U} - \frac{v\,dv^2}{V}\right] = \frac{u-v}{4}\left[uP^2 - vQ^2\right];$$

then, calling ϕ the inclination of the Darboux line to the curve $v = \text{constant}$, we have

$$\tan\phi = h\,\frac{Q}{P}, \text{ where } h \equiv \sqrt{\frac{u}{v}}. \tag{13}$$

From the equation of the Darboux lines

$$(14)\qquad \frac{Q}{P}=\sqrt{\frac{u-\varkappa}{v-\varkappa}}.$$

Combining (13) and (14), we find

$$\frac{v\tan^2\phi}{h^2}-u=\varkappa\left[\frac{\tan^2\phi}{h^2}-1\right].$$

Suppose that for the circular sections we write $\phi\equiv\lambda$, then since $\varkappa=0$,

$$\frac{v\tan^2\lambda}{h^2}=u,$$

and

$$(15)\qquad \varkappa=\frac{v\left[\tan^2\phi-\tan^2\lambda\right]}{\tan^2\phi-h^2}.$$

When $\phi=0$ or 90°, $\varkappa=u$ or v, as should be. Formula (15) enables us to calculate $\varkappa$ for any given point and any given direction.

8. The results of §2 can be arrived at in the following manner: If the coordinates x, y, z of the surface on which the Darboux lines lie be given in terms of two parameters u and v, equation (1) takes a very long form, but one which simplifies when the u and v are taken as the parameters of the lines of curvature on the surface. In this latter case we have

$$(16)\qquad \begin{aligned}
&3\,[E-\rho D]\,du\,d^2u+3\,[G-D''\rho]\,dv\,d^2v\\
&=\left\{\rho\left[\frac{\partial D}{\partial u}+\tfrac{1}{2}\,\frac{D}{E}\,\frac{\partial E}{\partial v}\right]-\tfrac{3}{2}\,\frac{\partial E}{\partial u}\right\}du^3\\
&+\left\{\rho\left[\frac{\partial D}{\partial v}-\tfrac{1}{2}\,\frac{\partial E}{\partial v}\left(\frac{D''}{G}-2\,\frac{D}{E}\right)\right]-\tfrac{3}{2}\,\frac{\partial E}{\partial v}\right\}du^2\,dv\\
&+\left\{\rho\left[\frac{\partial D''}{\partial u}-\tfrac{1}{2}\,\frac{\partial G}{\partial u}\left(\frac{D}{E}-2\,\frac{D''}{G}\right)\right]-\tfrac{3}{2}\,\frac{\partial G}{\partial u}\right\}du\,dv^2\\
&+\left\{\rho\left[\frac{\partial D''}{\partial v}+\tfrac{1}{2}\,\frac{D''}{G}\,\frac{\partial G}{\partial v}\right]-\tfrac{3}{2}\,\frac{\partial G}{\partial v}\right\}dv^3,
\end{aligned}$$

the symmetry of which is remarkable. If the lines $v=$ const. satisfy (16), we must have

$$\frac{\partial D}{\partial u}-\frac{D}{E}\,\frac{\partial E}{\partial u}=\frac{\partial\frac{1}{\rho_1}}{\partial u}=0,$$

where ρ_1 is one of the principal radii of curvature at the point considered. Then

$$\rho_1=f_1(v).$$

Similarly, when the lines $u=$ const. satisfy (16) we have

$$\rho_2=f_2(u).$$

Reasoning just as in §2, we find again the result there obtained. Most of the results obtained by Darboux and Enneper follow immediately when the equation of the Darboux lines is expressed in terms of u, v coordinates.

BALTIMORE, *March* 18, 1898.

Sur l'intégration hydraulique des équations différentielles.

Par M. Michel Petrovitch à *Belgrade* (*Serbie*).

1. Tous les intégraphes et les appareils pour l'intégration graphique des équations différentielles, proposés jusqu' aujourd'hui, sont fondé sur l'emploi de certains principes cinematiques, p. ex. sur les propriétés des roulettes. On en trouvera la liste et la description dans le Catalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente, von Walther Dyck, München 1892–1893.

Je me propose de montrer ici brièvement comment de telles intégrations peuvent se faire à l'aide des principes d'une nature tout-à-fait autre, facile à réaliser pratiquement, conduisant à des appareils très simples et pouvant intégrer des types assez généraux d'équations différentielles du premier ordre.

Supposons que l'on fasse immerger un corps solide M (Fig. 1) plus ou moins

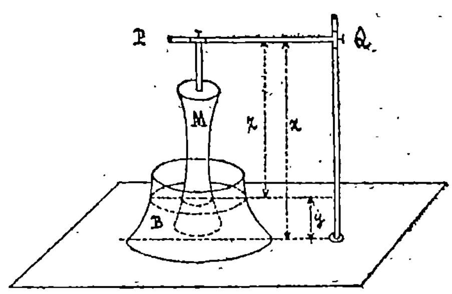


fig. 1.

profondément dans le liquide contenu dans un vase B. Le niveau du liquide montera ou s'abaissera d'après une certaine loi dépendant de la forme du corps M et du vase B et ces formes une fois fixées, la variation de la hauteur du niveau y, comptée à partir d'un plan horizontal fixe, p. ex. à partir de la face inférieure du vase B ne dépendra que de la distance x entre l'extrémité e de la tige ef et la face inférieure du vase B.

Désignons par z la distance entre le niveau du liquide et le plan PQ et soient $\Phi(y)$ et $F(z)$ les aires des sections horizontales du vase B à la hauteur y au-dessus de sa face inférieure et du corps M à la hauteur z comptée à partir du plan PQ. Les fonctions Φ et F dépendent de la forme du vase B et du corps M et ces formes une fois fixées, ces fonctions seront bien détérminées.

On obtiendra la relation entre x et y de la manière suivante.

En faisant immerger le corps M de sorte que x se change en $x-dx$ et y en $y+dy$, le volume du liquide qui s'est élevé au-dessus du niveau y sera

$$[\Phi(y)-F(z)]\,dy.$$

Ce volume est égal au volume du liquide déplacé par le corps M quand celui-ci sera immergé de dz, c'est-à-dire à

$$F(z)\,dz.$$

On en tire l'équation

$$[\Phi(y)-F(z)]\,dy = F(z)\,dz \tag{1}$$

et comme l'on a à chaque instant

$$z = x - y. \tag{2}$$

on aura l'équation

$$\Phi(y)\frac{dy}{dx} = F(x-y). \tag{3}$$

C'est l'équation différentielle du problème. En l'intégrant on aura la relation entre les variables x et y. Le rôle de la constante d'intégration joue la hauteur initiale du niveau.

Sur ce principe simple on peut fonder une méthode d'intégration graphique des certains types d'équations différentielles du premier ordre. On conçoit aussi la possibilité de construire plusieurs éspèces de nouveaux intégraphes de constructions simples, où des appareils pour le tracé continu des diverses courbes algébriques ou transcendantes.

Remarquons que dans la pratique il est le plus commode de donner au vase B et au corps M des formes cylindriques (Figs. 2 et 3) ayant deux faces planes

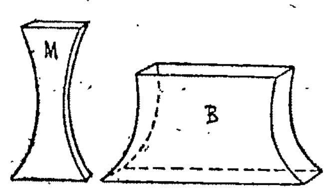


fig. 2. fig. 3.

et parallèles au plan de la figure, deux autres faces courbes, cylindriques et per-

pendiculaires à ce plan, et la face inférieure plane et horizontale. On aura alors

$$\left.\begin{aligned}\Phi(y) &= \alpha\phi(y),\\ F(z) &= \beta\theta(z)\end{aligned}\right\} \tag{4}$$

où α et β désignent les distances des faces parallèles du vase B et du corps M; $\phi(y)$ et $\theta(z)$ désignent leurs largeurs aux hauteurs respectives y et z.

On peut ainsi réaliser de telles formes des fonctions ϕ et θ que l'on voudra; sur une plaque p. ex. métalique on tracera les courbes correspondantes, ayant comme ordonnées horizontales respectives $\phi(y)$ et $\theta(z)$, à l'aide desquelles on formera facilement le vase B et le corps M.

L'équation différentielle deviendra

$$\alpha\phi(y)\frac{dy}{dx} = \beta\theta(x-y). \tag{5}$$

On peut faire varier la distance x de diverses manières, dont je n'indiquerai ici que les plus simples.

2. Imaginons p. ex. un cylindre vertical E (Fig. 4), tournant autour son axe

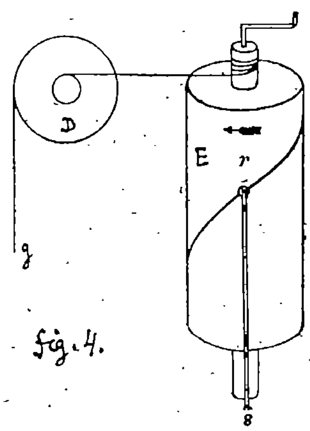


fig. 4.

et une poulie D tournant autour de son axe horizontal, perpendiculaire au plan de figure. Supposons le cylindre et la poulie liés par un fil de manière que si l'extrémité g du fil se meut du haut en bas, le cylindre tourne dans le sens indiqué par la flêche, les chemins parcourrus par g et un point quelconque de l'énveloppe du cylindre étant égaux.

A l'extrémité g est fixé le corps solide M. A l'extrémité s d'une autre tige rs, ne pouvant aussi que glisser verticalement, se trouve fixé un flotteur qui fera monter ou descendre la tige à mésure que le niveau du liquide dans le vase B

monte ou descend. Enfin, à l'extrémité r de la même tige fixons un crayon qui va tracer la courbe intégrale sur l'enveloppe du cylindre.

En faisant immerger le corps M plus ou moins profondément dans le liquide contenu dans le vase B, le niveau montera ou s'abaissera d'une manière continue et le crayon r tracera sur le cylindre la courbe intégrale de l'équation

$$\alpha\phi(y)\frac{dy}{dx} = \beta\theta(x-y),$$

où x et y seront l'abscisse et l'ordonnée d'un point quelconque de cette courbe. La courbe ainsi obtenue sera l'intégrale particulière de cette équation prenant pour $x=h$ la valeur $y=k$, h et k désignant les valeurs initiales de x et de la hauteur de niveau au-dessus du plan RS.

L'appareil servira donc pour l'intégration graphique des équations différentielles de la forme

$$\frac{dy}{dx} = f(y)\psi(x-y); \tag{6}$$

on donnera pour cela au vase B une forme telle qu'on ait

$$\phi(y) = \frac{1}{\alpha f(y)}$$

et au corps solide M une forme telle qu'on ait

$$\theta(z) = \frac{1}{\beta}\psi(z),$$

où α et β désignent les largeurs respectives de B et de M.

Si les rayons de la poulie D et du cylindre E n'étaient pas égaux, l'appareil servirait à l'intégration graphique des équations de la forme

$$\frac{dy}{dx} = f(y)\psi(ax-y). \tag{7}$$

Envisageons deux cas particulièrement simples qui peuvent se présenter.

1°. Si le corps M est prismatique, de sorte qu'on ait

$$\theta(z) = \text{const.} = \beta',$$

on aurait

$$x = \frac{\alpha}{\beta\beta'}\int\phi(y)\,dy$$

et l'appareil servirait comme intégraphe pour la courbe d'intersection du vase B avec le plan de figure.

2°. Si le vase B est prismatique de sorte qu'on ait

$$\phi(y) = \text{const.} = \alpha',$$

l'appareil construira les courbes intégrales des équations de la forme

$$\frac{dy}{dx} = \psi(x - y).$$

En y posant $$x - y = z,$$
l'équation devient $$\frac{dz}{dx} = 1 - \psi(z).$$
d'où $$x = \int \frac{dz}{1 - \psi(z)}$$

et l'appareil servira aussi dans ce cas comme intégraphe. Mais comme le crayon r décrit la courbe (x, y), cette courbe une fois construite, pour avoir z, correspondant à une valeur donnée de x, on retranchera y de x.

3. Au lieu de la disposition du paragraphe précédent, imaginons deux cylindres verticaux D et E de même diamétre, liés entre eux de manière (Fig. 5) que

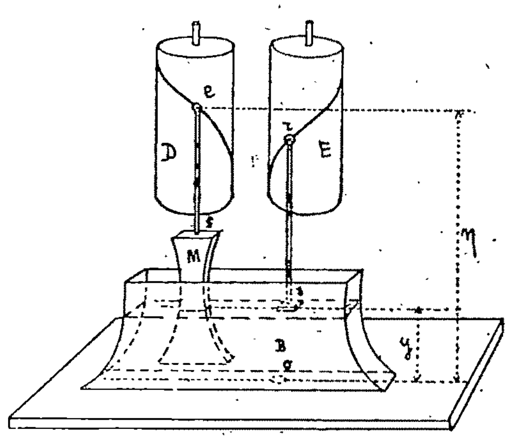


Fig. 5.

si le cylindre E se meut autour de son axe, le cylindre D le fait aussi. Fixons le corps solide M à l'extrémité f de la tige ef ne pouvant se déplacer que verticalement et ayant à son extrémité supérieure un clou métalique, qui touche à chaque instant l'enveloppe du cylindre D. A l'extrémité s d'une autre tige rs ne pouvant également se mouvoir que verticalement, fixons un flotteur qui fera monter ou descendre la tige à mésure que le niveau du liquide dans le vase B monte ou descend. Enfin, à l'extrémité r de la même tige fixons un crayon qui va tracer la courbe intégrale sur l'enveloppe du cylindre.

Supposons que sur le cylindre D soit enroulé un papier, sur lequel est tracée la courbe

$$\eta = f(\xi),$$

l'abscisse ξ étant comptée le long de la périphérie de la base du cylindre et l'ordonnée η le long des génératrices, à partir du plan fixe de la base du vase B.

En faisant tourner les cylindres p. ex. à l'aide d'une manivelle et assujettissant l'extrémité e de la tige ef à se trouver à chaque instant sur la courbe

$$\eta = f(\xi)$$

(par exemple en la guidant par la main, à mésure que la cylindre tourne), on aura à chaque instant

$$x = \eta = f(\xi)$$

et la hauteur du niveau, considérée comme fonction de ξ, sera donnée par l'intégration de l'équation différentielle

$$\alpha\phi(y)\frac{dy}{dx} = \beta\theta\,[f(\xi) - y]\,f'(\xi), \tag{8}$$

où l'intégrale pour $\xi =$ valeur initiale de l'abscisse, p. ex. $\xi = 0$, doit avoir la valeur $y =$ valeur initiale k de la hauteur du niveau, qui joue le rôle de la constante d'intégration. Cette intégrale sera donc tracée sur l'enveloppe du cylindre E par le crayon r.

Si p. ex. le corps M est prismatique de sorte qu'on ait

$$\theta(z) = \text{const.} = a,$$

la courbe décrite par le crayon r sera l'intégrale de l'équation

$$\alpha\phi(y)\frac{dy}{dx} = a\beta f'(\xi),$$

d'où

$$\frac{a\beta}{\alpha}\left[f(\xi) - f(0)\right] = \int_{k}^{y}\phi(y)\,dy.$$

En donnant au vase B et à la courbe $\eta = f(\xi)$ des formes convenables, l'appareil pourra servir à éffectuer le tracé continu de la courbe

$$y = \Psi\,[f(\xi)],$$

où Ψ est une fonction donnée à l'avance, lorsque la courbe $\eta = f(\xi)$ est construite etc.

Il est facile à voir que l'appareil peut servir de diverses manières comme intégraphe.

4. Concevons le même appareil que celui décrit dans le paragraphe précédent, mais avec les modifications suivantes:

1°. Les cylindres D et E tournent par action d'un mécanisme d'horlogerie, autour de leurs axes verticaux avec une vitesse uniforme, de manière qu'un point des leurs enveloppes respectives parcoure l'unité de longueur pendant l'unité de temps.

2°. Le liquide contenu dans le vase B s'écoule continuellement à travers un orifice pratiqué sur la face inférieure du vase B, dont on peut régler la largeur à volonté.

Le crayon r décrira alors sur l'enveloppe du cylindre E certaine courbe, dont on aura l'équation différentielle de la manière suivante. Si dans l'intervale de temps dt on fait immerger le corps M de sorte que x se change en $x - dx$ et y en $y + dy$, la quantité du liquide qui s'est élevée au-dessus du niveau y sera

$$[\alpha\phi(y) - \beta\theta(z)]\,dy$$

et cette quantité est égale à la différence de la quantité du liquide déplacé par le corps M quand celui-ci sera immergé de dz et celle qui s'est écoulée par l'orifice pendant le temps dt, c'est-à-dire à la différence

$$\beta\theta(z)\,dz - \lambda\sqrt{y}\,dt,$$

où

$$\lambda = \mu\Omega\sqrt{2g}$$

(μ étant le coefficient de contraction du liquide, Ω l'aire de l'orifice O et g la constante de gravitation). On en tire l'équation différentielle

$$[\alpha\phi(y) - \beta\theta(z)]\,dy = \beta\theta(z)\,dz - \lambda\sqrt{y}\,dt$$

et comme l'on a à chaque instant

$$z = x - y = f(t) - y,$$

l'équation différentielle du problème sera

$$\alpha\phi(y)\frac{dy}{dt} + \lambda\sqrt{y} - af'(t) = 0.$$

L'intégrale de cette équation, qui pour $t = 0$ prend la valeur $y = h$, égale à la valeur initiale de la hauteur du niveau, représente la loi de variation de cette hauteur avec le temps. Le crayon r tracera la courbe intégrale sur l'enveloppe du cylindre E.

On a ainsi l'intégration graphique des équations différentielles de la forme

$$\frac{dy}{dx} + F(y) = \Phi(y)\Psi(x)$$

(où F, Φ, Ψ sont des fonctions positives dans l'intervale considéré des variables) et de celles qui s'en déduisent par les changements de la forme

$$t = \Psi(\xi), \quad y = \theta(u);$$

il n'y a pour cela qu'à choisir convenablement les fonctions $\phi(y)$, $\theta(z)$ et $f(t)$, c'est-à-dire la forme du vase B, du corps M et de la courbe tracée sur le cylindre D.

En donnant p. ex. au vase B une forme telle qu'on ait

$$\alpha\phi(y) = \frac{1}{4\sqrt[4]{y}}$$

et au corps M une forme telle qu'on ait

$$\beta\theta(z) = \text{const.} = a$$

et en traçant sur le cylindre D la courbe correspondant à

$$f(t) = a\int \chi(t)\,dt,$$

la courbe (y, t) tracée par le crayon r sur le cylindre E, sera telle qu'à chaque instant la valeur $\sqrt[4]{y(t)}$ est égale à la valeur correspondante de l'intégrale $u(t)$ de l'équation de Riccati

$$\frac{du}{dt}\,\chi(t) - \lambda u^2,$$

qui pour $t = 0$ prend la valeur $\sqrt[4]{h}$, h étant la hauteur initiale du niveau.

Des principes analogues s'appliquent à bien d'autres types d'équations. On aurait des types nouveaux d'équations intégrables graphiquement en faisant p. ex. varier l'aire de l'ouverture O suivant les lois données, ce qui est facile à faire à l'aide d'un troisième cylindre à l'axe horizontal, sur lequel serait tracée la courbe correspondant à la loi donnée.

On the Hyperelliptic Sigma Functions.

BY H. F. BAKER, *St. John's College, Cambridge.*

INTRODUCTION.

It is known of what importance for the development of the theory of elliptic functions is the equation

$$\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2(u)\,\sigma^2(v)} = \frac{d^2}{du^2}\log\sigma u - \frac{d^2}{dv^2}\log\sigma v.$$

It becomes therefore a matter of interest to enquire whether an analogous equation holds for higher cases. Denoting a theta function with half-integer characteristic A by $\vartheta(u;\,A)$, and a theta function whose characteristic is the sum of the half-integer characteristics A, B, by $\vartheta(u;\,AB)$, it is known that there exist, in the general case of Abelian functions, $\frac{1}{2}p(p+1)$ equations of the form

$$\frac{\partial^2}{\partial u_i\,\partial u_j}\log\vartheta(u;\,A) = \sum_r C_r\frac{\vartheta^2(u;\,AP_r)}{\vartheta^2(u;\,A)},$$

wherein A is an arbitrary characteristic, P_r is one of a group of 2^p characteristics over which the summation on the right side is to extend, C_r is a constant, and u_i, u_j are any two of the p arguments $u_1, \ldots, u_p$ (Frobenius, Crelle LXXXIX, 1880). These equations furnish $\frac{1}{2}p(p+1)$ conditions for the expression of the 2^p-1 quotients $\vartheta^2(u;\,AP_r)/\vartheta^2(u;\,A)$ in terms of functions of the form $\frac{\partial^2}{\partial u_i\,\partial u_j}\log\vartheta(u;\,A)$. If these equations were capable of solution, we might expect, by means of the addition equation of the theta functions, to be able to obtain the function $\vartheta(u+v;\,A)\,\vartheta(u-v;\,A)/\vartheta^2(u;\,A)\,\vartheta^2(v;\,A)$ in terms of functions $\frac{\partial^2}{\partial u_i\,\partial u_j}\log\vartheta(u;\,A)$. Except in case $p=2$, in which $\frac{1}{2}p(p+1)=3=2^p-1$,

it is not however obvious* that the equations are capable of solution. It is therefore of interest, and it is the final (though not the only) purpose of this paper to prove, that *in the hyperelliptic case, for any value of* p, *the required expression can be effected.* To be more precise, it is shown below, that *if* Q_i *be any one of a certain group of* 2^p *characteristics, then, in the hyperelliptic case*

$$\frac{\vartheta(u+v;\ Q_i)\,\vartheta(u-v;\ Q_i)}{\vartheta^2(u;\ Q_i)\,\vartheta(v;\ Q_i)}$$

can be expressed as an integral polynomial in the $p\,(p+1)$ *functions*

$$\frac{\partial^2}{\partial u_i\,\partial u_j}\log\vartheta(u;\ Q_i),\qquad \frac{\partial^2}{\partial v_i\,\partial v_j}\log\vartheta(v;\ Q_i).$$

I have desired to arrive at this result, starting only from the general results associated with Riemann's theory of Abelian functions, and without assuming properties for which, though in some cases their general character is already well known, I was not able to give a precise reference to the existing literature. In some cases, to save space, I have ventured to give references to my recent book on Abelian functions; these references are printed thus, [B. 25]. An analysis of the steps of the argument which leads to the final result, is prefixed to the development. The formulæ in Nos. VII and VIII of this analysis would also appear to be of importance in the theory of the hyperelliptic theta functions.

ANALYSIS.

I.—*Of the method of the paper.*

We suppose the fundamental algebraical equation to be

$$y^2 = 4P(x)\,Q(x),$$

wherein

$$Q(x) = (x-c_1)(x-c_2)\ldots\ldots(x-c_p)(x-c),$$
$$P(x) = (x-a_1)(x-a_2)\ldots\ldots(x-a_p),$$

that is, we suppose one of the branch places to be at infinity. We suppose the cross lines of the Riemann surface, by which we pass from one sheet to the other, to be c_1a_1, c_2a_2, ..., c_pa_p, ca, where a denotes the branch place at

*Apart from the general theorems announced by Weierstrass, Crelle, LXXXIX, 1880, p. 7, from which such a result was to be expected.

infinity, and we call $c_1, a_1, c_2, a_2, \ldots c_p, a_p, c, a$, the *ascending order* of the branch places; we do not however suppose the quantities c_1, a_1, etc., to be real The dissection of the surface, whereby it is changed into the p-ply connected surface on which the Abelian integrals are single valued, is taken to be that denoted by the annexed diagram. Other methods of dissection are discussed below; from the discussion the reasons for the adoption of this method will appear.

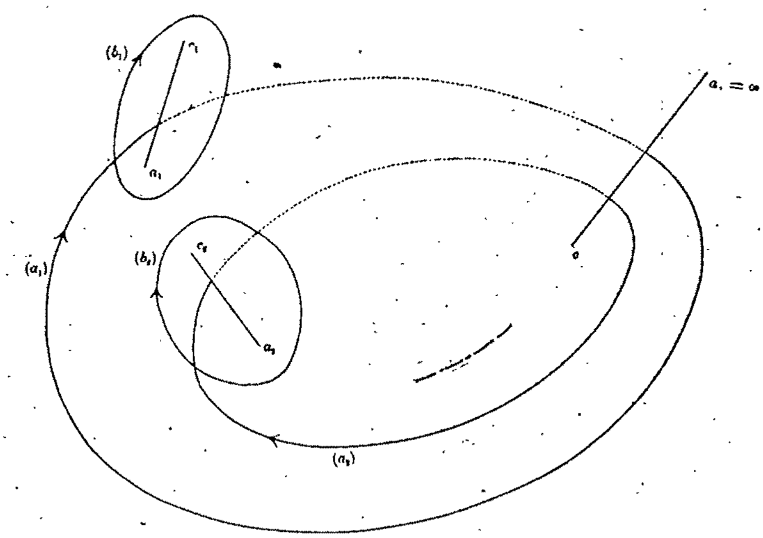


FIG. A.

[The period of any Abelian integral at a period loop is the constant by which the value of the integral on the side of the loop which is on the left when the loop is described in the direction of the arrow-head exceeds the value of the integral on the other side. The loops themselves are called by the letters (b_1), (a_1), ... here placed in brackets. The loops (a_1), (a_2), ... are called the first, second, ... period loops of the first kind, the loops (b_1), (b_2), ... are called the first, second, ... period loops of the second kind. In the figure $p - 2$ pairs of loops are not shown.]

The finite branch places, taken in the ascending order, are frequently denoted, respectively, by $b_1, b_2, b_3, b_4, \ldots b_{2p-1}, b_{2p}, b_{2p+1}$. In some cases, however, where no misunderstanding can arise, a batch of k finite branch places, taken in the ascending order but not necessarily beginning with the branch place c_1, is denoted by $b_1, b_2, \ldots, b_k$.

It is known that any one of the 2^{2p} theta functions arising from the fundamental algebraic equation

$$y^2 = (x, 1)_{2p+2} = f(x)$$

is associated with a decomposition of the integral polynomial $f(x)$ into two factors, in the form

$$f(x) = \phi_x^{p+1-2\mu} \psi_x^{p+1+2\mu};$$

in what follows we are primarily, though not exclusively, concerned with those 2^p functions which arise for all decompositions in which $\psi_x^{p+1+2\mu}$ has a fixed factor of order $p+1$, namely, the factor which we have denoted by $Q(x)$. Cf. below, No. VII of the Analysis.

II.—*Of two signs depending on the dissection of the Riemann surface.*

If

$$Q = \tfrac{1}{2}\binom{q'}{q} = \tfrac{1}{2}\begin{pmatrix} q_1', q_2', \ldots, q_p' \\ q_1, q_2, \ldots, q_p \end{pmatrix}, \quad K = \tfrac{1}{2}\binom{k'}{k} = \tfrac{1}{2}\begin{pmatrix} k_1', k_2', \ldots, k_p' \\ k_1, k_2, \ldots, k_p \end{pmatrix}$$

be two characteristics of half-integers, we use (after Frobenius) the abbreviations

$$|Q, K| = \sum_{r=1}^{p} (q_r k_r' - q_r' k_r), \quad \binom{Q}{K} = e^{\pi i \sum_{r=1}^{p} q_r' k_r}.$$

Further, b denoting any one of the $2p+1$ finite branch places, and $2\omega_{r,i}$, $2\omega_{r,i}'$ denoting the periods of an integral of the first kind, $u_r^{x,a}$, at the i-th period loops respectively of the first and second kind, if

$$u_r^{a,b} = \beta_1\omega_{r,1} + \ldots + \beta_p\omega_{r,p} + \beta_1'\omega_{r,1}' + \ldots + \beta_p'\omega_{r,p}',$$

(so that, as is known, $\beta_1, \ldots, \beta_p'$ are integers), we put

$$B = \tfrac{1}{2}\binom{\beta'}{\beta}.$$

In regard then to the $2p+1$ half-integer characteristics, B, thus arising, it is important to consider the values of the two quantities

$$e^{4\pi i |B_i, B_j|}, \quad \binom{B_i}{B_j},$$

wherein B_i, B_j are any two of the $2p+1$ characteristics in question. The origin

of these two quantities will be seen in the two following Nos. III, IV respectively. These signs may be replaced by the single one

$$\sqrt{\left(\frac{Q}{K}\right)} = e^{\frac{1}{2}\pi i \sum_{r=1}^{p} q'_r k_r};$$

but this is practically not very convenient.

III.—*On the fundamental radical functions.*

On the p-ply connected dissected Riemann surface upon which the Abelian integrals are single valued, there exist $2p+1$ single valued functions, of which the squares are the $2p+1$ functions $x-b$, b denoting any one of the finite branch places. Supposing the values of y to be beforehand allocated to the places of the Riemann surface, they shall be defined by the facts, (i) that at infinity the ratio of any two of them is unity, (ii) that their product is equal to $\frac{1}{2}y$. We denote them by the symbols $\sqrt{x-b}$. We are then able to define the symbols $\sqrt{b_i-b_j}$ as the value of $\sqrt{x-b_j}$ at the place b_i. With this definition we are able to prove the equation

$$\sqrt{b_i-b_j} = \sqrt{b_j-b_i} \,.\, e^{\frac{1}{2}\pi i \,|\, B_i,\, B_j \,|},$$

which holds for any method of dissection of the Riemann surface.

IV.—*Of the expression of a certain theta-quotient.*

If $u_1^{x,a}, \ldots\ldots, u_p^{x,a}$ be a system of linearly independent integrals of the first kind, and, as usual, $a_1, \ldots\ldots, a_p$ being the branch places before described,

$$u_r = u_r^{x_1,a_1} + \ldots\ldots + u_r^{x_p,a_p}, \qquad (r = 1, 2, \ldots\ldots, p)$$

then the theta quotient

$$\frac{\vartheta^2(u \,|\, B_i B_j)\, \vartheta^2(u)}{\vartheta^2(u \,|\, B_i)\, \vartheta^2(u \,|\, B_j)}$$

is equal to

$$\left(\frac{B_i}{B_j}\right)(b_i-b_j)\left\{\frac{1}{2}\sum_{r=1}^{p}\frac{y_r}{(x_r-b_i)(x_r-b_j)\,F'(x_r)}\right\}^2,$$

where

$$F(x) = (x-x_1)(x-x_2)\ldots\ldots(x-x_p).$$

(Cf., for instance, Bolza, American Journal, XVII (1895), where, however, the sign of the right side is not made precise.)

V.—*Of the dissection of the Riemann surface.*

It is possible to take the period loops, on the Riemann surface, so that, *when* $i > j$, the value of each of the quantities $|B_i, B_j|$, $\left(\begin{smallmatrix} B_i \\ B_j \end{smallmatrix}\right)$ shall be independent of i and j. This is manifestly a convenience. These two quantities are however independent; it is possible to choose dissections in which their values form any one of the four combinations $(+1, +1)$, $(-1, -1)$, $(+1, -1)$, $(-1, +1)$. We adopt as our standard method of dissection one in which *when* $i > j$,

$$|B_i, B_j| = -|B_j, B_i| = +1,$$

$$\binom{B_i}{B_j} = -\binom{B_j}{B_i} = +1.$$

Then we have, when $i > j$,

$$\sqrt{b_j - b_i} = -i\sqrt{b_i - b_j},$$

$$\frac{\vartheta^2(u \mid B_i B_j)\, \vartheta^2(u)}{\vartheta^2(u \mid B_i)\, \vartheta^2(u \mid B_j)} = (b_i - b_j)\left\{\tfrac{1}{2}\sum_{r=1}^{p} \frac{y_r}{(x_r - b_i)(x_r - b_j)\, F'(x_r)}\right\}^2.$$

VI.—*Resulting preliminary formulæ. Introduction of sigma-functions.*

Let

$$u_r^{x,a} = \int_a^x \frac{x^{r-1}dx}{y},$$

and let the value, when the arguments $u_1, \ldots, u_p$ are zero, of the expression

$$\frac{1}{\vartheta(0)}\, \frac{1}{i\sqrt{f'(a_r)/4}} \left(\frac{\partial}{\partial u_1} + a_r \frac{\partial}{\partial u_2} + \ldots + a_r^{p-1} \frac{\partial}{\partial u_p}\right) \vartheta(u \mid A_r),$$

wherein A_r is the half-integer characteristic associated with the half-period u^{a, a_r}, be denoted by λ_r; herein, as always in what follows, $i\sqrt{f'(a_r)/4}$ is written to denote

$$i\sqrt{a_r - c_1}\sqrt{a_r - a_1}\sqrt{a_r - c_2} \ldots \sqrt{a_r - c},$$

the symbols $\sqrt{a_r - c_1}$, etc., being as in No. III; then, for a proper determination of the sign of the denominator, we have

$$\lambda_r = \frac{(-1)^{p-r}\sqrt{P'(a_r)}}{(i\sqrt{f'(a_r)/4})^{\frac{1}{2}}} = \frac{\sqrt{(-1)^{p-r}P'(a_r)}}{(\sqrt{(-1)^{2p-2r+1}f'(a_r)/4})^{\frac{1}{2}}},$$

where $\sqrt{P'(a_r)}$ denotes

$$\sqrt{a_r - a_1}\sqrt{a_r - a_2}\ldots\ldots\sqrt{a_r - a_p},$$

the symbols $\sqrt{a_r - a_1}$, etc., being as in No. III, and $\sqrt{(-1)^{p-r}P'(a_r)}$ denotes

$$\sqrt{a_r - a_1}\ldots\ldots\sqrt{a_r - a_{r-1}}\sqrt{a_{r+1} - a_r}\ldots\ldots\sqrt{a_p - a_r}.$$

Further we have

$$\frac{\vartheta(u\,|\,A_r)}{\vartheta(u)} = i\,\frac{\sqrt{(-1)^{p-r+1}F(a_r)}}{(\sqrt{(-1)^{2p-2r+1}f'(a_r)/4})^{\frac{1}{2}}} = \frac{\sqrt{F(a_r)}}{(i\sqrt{f'(a_r)/4})^{\frac{1}{2}}} = (-1)^{p-r}\lambda_r\,\frac{\sqrt{F(a_r)}}{\sqrt{P'(a_r)}},$$

where

$$\sqrt{(-1)^{p-r+1}F(a_r)} = \sqrt{a_r - x_1}\ldots\ldots\sqrt{a_r - x_{r-1}}\sqrt{x_r - a_r}\ldots\ldots\sqrt{x_p - a_r},$$
$$\sqrt{F(a_r)} = \sqrt{a_r - x_1}\ldots\ldots\sqrt{a_r - x_p};$$

we have previously (No. III) defined the sense in which the symbols $\sqrt{x - b}$ are used; it is convenient also, as here, to introduce symbols denoted by $\sqrt{b - x}$; we *define* then the symbol $\sqrt{b - x}$, wherein b is any one of the $2p + 1$ finite branch places, by the equation

$$\sqrt{x - b} = -i\sqrt{b - x}.$$

Such formulæ as these have of course been given before (e. g. Königsberger, Crelle, LXIV (1865), for the case when the branch values $c_1, a_1, c_2, \ldots\ldots$ are real); but, I think, without the precise determination of the signs of the square roots involved which is necessary for the deductions to be drawn in this paper.

Further, B_r, B_s, denoting as before the characteristics associated with the half-periods u^{a, b_r}, u^{a, b_s}, and B_rB_s denoting the sum of these characteristics, *without reduction*, we put

$$\frac{\vartheta(u\,|\,B_rB_s)\,\vartheta(u)}{\vartheta(u\,|\,B_r)\,\vartheta(u\,|\,B_s)} = \varepsilon_{r,s}\sqrt{b_r \sim b_s}\left\{\tfrac{1}{2}\sum_{i=1}^{p}\frac{y_i}{(x_i - b_r)(x_i - b_s)\,F'(x_i)}\right\},$$

where $b_r \sim b_s$ is written, instead of $b_r - b_s$, to denote that in the ascending order of the branch places, b_r has a higher place than b_s, and $\varepsilon_{r,s}$ is a square root of unity, in fact defined by this equation, which we do not determine.

Then we find, if $\zeta_{r,s} = \varepsilon_{r,s}\sqrt{b_r \sim b_s}$, a notation which will be frequently employed, that

$$\frac{\vartheta(u|B_rB_sB_t)\vartheta^3(u)}{\vartheta(u|B_r)\vartheta(u|B_s)\vartheta(u|B_t)} = -\zeta_{r,s}\zeta_{s,t}\zeta_{r,t}\left\{\tfrac{1}{2}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_r)(x_i-b_s)(x_i-b_t)F'(x_i)}\right\}.$$

It is to be noticed that $\zeta_{r,s} = \zeta_{s,r}$, etc., as follows from the definition.

We now suppose b_r, b_s, b_t to be chosen from the branch places $a_1, \ldots, a_p$; and we introduce the definitions

$$\sigma_r(u) = \lambda_r\vartheta(u|A_r)/\vartheta(0),$$
$$\sigma_{r,s}(u) = -\frac{\lambda_r\lambda_s}{\zeta_{r,s}}\vartheta(u|A_rA_s)/\vartheta(0),$$
$$\sigma_{r,s,t}(u) = +\frac{\lambda_r\lambda_s\lambda_t}{\zeta_{r,s}\zeta_{s,t}\zeta_{r,t}}\vartheta(u|A_rA_sA_t)/\vartheta(0),$$

wherein A_r is the characteristic associated with the half-period u^{a,a_r}, and A_rA_s denotes the sum of the characteristics A_r, A_s, without reduction, etc. Then, bearing in mind that the argument u_k is given by

$$u_k = \int_{a_1}^{x_1}\frac{x^{k-1}dx}{y} + \ldots + \int_{a_p}^{x_p}\frac{x^{k-1}dx}{y}, \qquad (k = 1, \ldots, p)$$

we find that the first terms in the expansion of $\sigma_r(u)$ in a series of integral positive powers of $u_1, \ldots, u_p$ are the linear terms which may symbolically be denoted by

$$\frac{P(\xi)}{\xi - a_r}, = (\xi-a_1)(\xi-a_2)\ldots(\xi-a_p),$$
$$= \xi^{p-1} + A_1\xi^{p-2} + A_2\xi^{p-3} + \ldots + A_{p-1}, \text{ say},$$

provided we replace the powers of the symbolical quantity ξ according to the definitions

$$\xi^{i-1} = u_i, \quad \xi^0 = u_1; \qquad (i = 1, 2, \ldots, p)$$

that is, the linear terms are

$$u_p + A_1u_{p-1} + A_2u_{p-2} + \ldots + A_{p-1}u_1.$$

Similarly the first terms in the expansion of $\sigma_{r,s}(u)$ are linear terms given, symbolically, by

$$\frac{P(\xi)}{(\xi-a_r)(\xi-a_s)}, = \xi^{p-2} + \xi^{p-3}B_1 + \xi^{p-4}B_2 + \ldots + B_{p-2}, \text{ say},$$
$$= u_{p-1} + B_1u_{p-2} + B_2u_{p-3} + \ldots + B_{p-2}u_1;$$

and the first terms in the expansion of $\sigma_{r,s,t}(u)$ are quadratic terms given symbolically by

$$\tfrac{1}{2}\phi(\xi)\,\phi(\xi')(\xi-\xi')^2,$$

where

$$\phi(\xi)=\frac{P(\xi)}{(\xi-a_r)(\xi-a_s)(\xi-a_t)},$$

and, after the division has been carried out, we are to replace the equivalent symbols ξ, ξ' according to the laws

$$\xi^{i-1}=u_i=\xi'^{i-1},\quad \xi^{\circ}=u_1=\xi'^{\circ}. \qquad (i=1,\,2,\,\ldots,\,p)$$

The function $\sigma_r(u)$ is one of p functions; the function $\sigma_{r,s}(u)$ is one of $\tfrac{1}{2}p\,(p-1)$ functions; the function $\sigma_{r,s,t}(u)$ is similarly one of $\tfrac{1}{6}p\,(p-1)(p-2)$ functions.

These expansions show that the functions agree with the hyperelliptic sigma functions considered by Klein. Cf. Burkhardt, Math. Annal., XXXII (1888), p. 442.

VII.—*New expression of theta functions of three or more suffixes in terms of functions of one or two suffixes.*

Let $b_1, b_2, \ldots, b_{2n+1}$ be any finite branch places, taken in any order; let

$$\phi_{r,s}(u)=\frac{\vartheta(u\,|\,B_rB_s)}{\zeta_{r,s}},$$

the notation being as before, and put

$$\nabla_{12\ldots k}=\zeta_{1,2}\zeta_{1,3}\zeta_{2,3}\zeta_{1,4}\zeta_{2,4}\zeta_{3,4}\ldots\zeta_{1,k}\zeta_{2,k}\ldots\zeta_{k-1,k};$$

let the suffixes $1, 2, \ldots, 2n$ be divided into two sets $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$, and put

$$D_n=(b_{r_1}-b_{r_2})\ldots(b_{r_1}-b_{r_n})(b_{r_2}-b_{r_3})\ldots(b_{r_2}-b_{r_n})\ldots(b_{r_{n-1}}-b_{r_n}),$$
$$E_n=(b_{s_1}-b_{s_2})\ldots(b_{s_1}-b_{s_n})(b_{s_2}-b_{s_3})\ldots(b_{s_2}-b_{s_n})\ldots(b_{s_{n-1}}-b_{s_n});$$

then, if $\vartheta_{1,2\ldots k}(u)$ denote $\vartheta(u\,|\,B_1B_2\ldots B_k)$, where $B_1\ldots B_k$ denotes the sum, without reduction, of the half-integer characteristics associated with the half-periods $u^{a,b_1}, \ldots, u^{a,b_k}$, we have the theorem

$$\vartheta^{n-1}(u)\,\vartheta_{12\ldots 2n}(u)=\begin{vmatrix}\phi_{r_1,s_1}, & \phi_{r_1,s_2}, & \ldots, & \phi_{r_1,s_n}\\ \phi_{r_2,s_1}, & \phi_{r_2,s_2}, & \ldots, & \phi_{r_2,s_n}\\ \ldots & \ldots & \ldots & \ldots\\ \phi_{r_n,s_1}, & \phi_{r_n,s_2}, & \ldots, & \phi_{r_n,s_n}\end{vmatrix}\frac{\nabla_{12\ldots 2n}}{D_nE_n}$$

Further, if the suffixes 1, 2, ..., $2n+1$ be divided into two sets $r_1, \ldots, r_n$ and $s_1, \ldots, s_{n+1}$, and D_n be defined as before, but

$$E_{n+1} = E_n (b_{s_1} - b_{s_{n+1}})(b_{s_2} - b_{s_{n+1}}) \ldots (b_{s_n} - b_{s_{n+1}}),$$

then we have the theorem

$$\vartheta_n(u)\,\vartheta_{12\ldots 2n+1}(u) = \begin{vmatrix} \vartheta_{s_1}, & \vartheta_{s_2}, & \ldots, & \vartheta_{s_{n+1}} \\ \phi_{r_1, s_1}, & \phi_{r_1, s_2}, & \ldots, & \phi_{r_1, s_{n+1}} \\ \ldots & \ldots & \ldots & \ldots \\ \phi_{r_n, s_1}, & \phi_{r_n, s_2}, & \ldots, & \phi_{r_n, s_{n+1}} \end{vmatrix} \frac{\nabla_{12\ldots 2n+1}}{D_n E_{n+1}}.$$

In these equations ϑ_{s_1} is written for $\vartheta(u \mid B_{s_1})$, and $\phi_{r,s}$ for $\phi_{r,s}(u)$.

We now suppose the branch places $b_1, \ldots, b_{2n+1}$ to be chosen from among the places $a_1, \ldots, a_p$; and we put

$$\begin{aligned} \sigma(u) &= \frac{\vartheta(u)}{\vartheta(0)}, \\ \sigma_{12\ldots 2n}(u) &= (-1)^n \frac{\lambda_1\lambda_2 \ldots \lambda_{2n}}{\nabla_{12\ldots 2n}} \frac{\vartheta(u \mid A_1 \ldots A_{2n})}{\vartheta(0)}, \\ \sigma_{12\ldots 2n+1}(u) &= \frac{\lambda_1\lambda_2 \ldots \lambda_{2n+1}}{\nabla_{12\ldots 2n+1}} \frac{\vartheta(u \mid A_1 \ldots A_{2n+1})}{\vartheta(0)}, \end{aligned}$$

wherein $A_1 \ldots A_{2n}$ denotes the sum of the characteristics associated with the half-periods $u^{a, a_1}, \ldots, u^{a, a_{2n}}$, and $2n+1$ is supposed not greater than p, so that we define, hereby, 2^p sigma functions. Then we find

$$\sigma^{(n-1)}(u)\,.\,\sigma_{12\ldots 2n}(u) = \begin{vmatrix} \sigma_{r_1, s_1}(u), & \ldots, & \sigma_{r_1, s_n}(u) \\ \ldots & \ldots & \ldots \\ \sigma_{r_n, s_1}(u), & \ldots, & \sigma_{r_n, s_n}(u) \end{vmatrix} \frac{1}{D_n E_n}.$$

and

$$\sigma^n(u)\,\sigma_{12\ldots(2n+1)}(u) = \begin{vmatrix} \sigma_{r_1, s_1}(u), & \ldots, & \sigma_{r_1, s_{n+1}}(u) \\ \ldots & \ldots & \ldots \\ \sigma_{r_n, s_1}(u), & \ldots, & \sigma_{r_n, s_{n+1}}(u) \\ \sigma_{s_1}(u) & , \ldots, & \sigma_{s_{n+1}}(u) \end{vmatrix} \frac{1}{D_n E_{n+1}}.$$

These expressions bring into evidence the fact that $\sigma_{12\ldots k}(u)$ vanishes to order $\frac{1}{2}k$, or $\frac{1}{2}(k+1)$, when the arguments $u_1, \ldots, u_p$ are zero, according as k is

even or odd; and they show that the first terms in the expansion of $\sigma_{12 \dots 2n}(u)$ in powers of $u_1, \dots, u_p$, are those given symbolically by

$$\frac{1}{n!}\,\phi(\xi_1)\dots\phi(\xi_n)\,\Delta(\xi_1,\dots,\xi_n),$$

where $\phi(\xi) = P(\xi)/(\xi - a_1)\dots(\xi - a_{2n})$, $\Delta(\xi_1,\dots,\xi_n)$ denotes the product of the squares of the differences of $\xi_1,\dots,\xi_n$, and, after the divisions have been carried out, we are to replace the powers of the equivalent symbols $\xi_1,\dots,\xi_n$ according to the law expressed by

$$\xi_r^{i-1} = u_i, \quad \xi_r^0 = u_1. \qquad \begin{pmatrix} r = 1, 2, \dots, n \\ i = 1, 2, \dots, p \end{pmatrix}$$

Similarly, with $\phi(\xi) = P(\xi)/(\xi - a_1)\dots(\xi - a_{2n+1})$, the first term in the expansion of $\sigma_{12 \dots 2n+1}(u)$ is given by

$$\frac{1}{(n+1)!}\,\phi(\xi_1)\dots\phi(\xi_{n+1})\,\Delta(\xi_1,\dots,\xi_{n+1}).$$

Cf. Burkhardt, Math. Annal., XXXII (1888), p. 442. The function $\sigma_{12 \dots 2n}(u)$ has the 'algebraical characteristic' given by

$$\phi_x^{p+1-2\mu} = (x-a)\,\frac{P(x)}{(x-a_1)\dots(x-a_{2n})},$$
$$\psi_x^{p+1+2\mu} = Q(x)\,.\,(x-a_1)\dots(x-a_{2n}),$$

and the function $\sigma_{12 \dots (2n+1)}(u)$ the algebraical characteristic given by

$$\phi_x^{p+1-2\mu} = \frac{P(x)}{(x-a_1)\dots(x-a_{2n+1})},$$
$$\psi_x^{p+1+2\mu} = (x-a)\,Q(x)\,(x-a_1)\dots(x-a_{2n+1}).$$

VIII.—*New expression for the square of a theta function of three or more suffixes in terms of the squares of functions of one and two suffixes.*

It is known that the skew symmetrical determinant of $2n$ rows and columns, of which the $(i, j)^{\text{th}}$ element $a_{i,j}$ is such that $a_{i,j} = -a_{j,i}$, $a_{i,i} = 0$, is the square of a rational polynomial of degree n in the elements of the determi-

nant; this polynomial—called by Cayley a Pfaffian—we denote by $(123 \ldots\ldots 2n)$; it may be defined by the equation

$$(12 \ldots\ldots 2n) = a_{12}(34 \ldots\ldots 2n) - a_{13}(245 \ldots\ldots 2n) + a_{14}(235 \ldots\ldots 2n) - \ldots\ldots + a_{1,2n}(23 \ldots\ldots \overline{2n-1}),$$

where $n = 2, 3, 4, \ldots\ldots$

With this notation we have the theorem expressed by the two following equations

$$\vartheta^{2(n-1)}(u) \cdot \vartheta^2_{12 \ldots\ldots 2n}(u) = (12 \ldots\ldots 2n),$$
$$\vartheta^{2n}(u) \cdot \vartheta^2_{12 \ldots\ldots (2n+1)}(u) = (012 \ldots\ldots 2n+1),$$

where, if $b_1, \ldots\ldots, b_{2n+1}$ be any finite branch places, and $B_1 B_2 \ldots\ldots B_k$ denote the sum (without reduction) of the characteristics associated with the half-periods $u^{a, b_1}, \ldots\ldots, u^{a, b_k}$,

$$\vartheta_{12 \ldots\ldots k}(u) = \vartheta(u \mid B_1 B_2 \ldots\ldots B_k),$$

as before, and

$$a_{ij} = \vartheta^2_{ij}(u) \text{ when } i < j, \quad a_{ij} = -\vartheta^2_{ij}(u) \text{ when } i > j,$$
$$a_{0,i} = \vartheta^2_i(u), \quad a_{i,0} = -\vartheta^2_i(u).$$

These equations enable us to express the quotient $\vartheta^2_{12 \ldots\ldots k}(u)/\vartheta^2(u)$, in which $k > 2$, as a rational integral polynomial in the $\frac{1}{2}p(p+1)$ quotients $\vartheta^2_i(u)/\vartheta^2(u)$, $\vartheta^2_{ij}(u)/\vartheta^2(u)$.

In the applications which are here to be made of these expressions the branch places $b_1, \ldots\ldots, b_{2n+1}$ will be chosen from among $a_1, \ldots\ldots, a_p$.

IX.—*On an addition equation for the hyperelliptic theta functions and the resulting proof of the expressions of the quotients* $\vartheta^2(u \mid A_i)/\vartheta^2(u)$, $\vartheta^2(u \mid A_i A_j)/\vartheta^2(u)$ *by means of functions* $\wp_{r,s}(u)$.

Let A_i denote the half-integer characteristic associated with the half-period u^{a, a_i}, and let Q denote any one of the group of 2^p characteristics formed by the addition of $0, 1, 2, \ldots\ldots, p$ of the characteristics $A_1, \ldots\ldots, A_p$. Further, let

$$\wp_{i,j}(u) = -\frac{\partial^2}{\partial u_i \partial u_j} \log \vartheta(u).$$

Then we have the addition equation (given by Königsberger, Crelle, LXIV)

$$\vartheta^2(0)\,\vartheta(u+v)\,\vartheta(u-v)=\sum_Q \vartheta^2(u\,|\,Q)\,\vartheta^2(v\,|\,Q).$$

Now suppose that in this equation, for small values of the arguments v, both sides are expanded in powers of these arguments, and the coefficients of the quadratic powers of these arguments on the two sides of the equation are equated to one another. As follows from No. VII, the only terms on the right side wherein the lowest powers of the arguments v are not of higher degree than the second are those involving functions of one and two suffixes. The left side is equal to

$$\vartheta^2(0)\,\vartheta^2(u)\left[1-\sum_{i=1}^{p}\sum_{j=1}^{p} v_i v_j \wp_{ij}(u)+\ldots\right].$$

Hence we obtain $\frac{1}{2}p(p+1)$ equations whereby the $\frac{1}{2}p(p+1)$ quotients $\vartheta^2(u\,|\,A_i)/\vartheta^2(u)$, $\vartheta^2(u\,|\,A_iA_j)/\vartheta^2(u)$ are expressible linearly by the functions $\wp_{ij}(u)$.

Utilizing the results of No. VI to solve these equations, we find, if

$$\frac{\vartheta(u)}{\vartheta(0)}=1+\tfrac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} c_{ij}\,u_i u_j+\ldots,$$

that

$$\frac{\vartheta^2(u\,|\,A_r)}{\vartheta^2(u)}=-\frac{1}{M_r}\sum_{i=1}^{p}\left[c_{p,i}+\wp_{p,i}(u)\right]a_r^{i-1},$$

$$\frac{\vartheta^2(u\,|\,A_rA_s)}{\vartheta^2(u)}=\frac{a_r\sim a_s}{M_rM_s}\sum_{i=1}^{p}\sum_{j=1}^{p}\left[c_{i,j}+\wp_{i,j}(u)\right]a_r^{i-1}a_s^{j-1},$$

where, with the meaning explained in No. VI,

$$M_r=i\sqrt{f'(a_r)}/4, \qquad (r=1,2,\ldots,p)$$

and $a_r\sim a_s$ is written instead of a_r-a_s to indicate that in the ascending order of the branch places $a_1,\ldots,a_p$, a_r has a higher place than a_s, or in other words, that $r>s$.

Results of this form have been given before. See, for instance, Wiltheiss, Math. Annal., XXXI (1888), p. 417, and Bolza, American Journal, XVII (1895), and the references to Brioschi and others there given.

In these equations, and in the previous Nos. of this Analysis, the function $\vartheta(u)$ is that given by

$$\sum_{n_1=-\infty}^{\infty}\cdots\sum_{n_p=-\infty}^{\infty} e^{au^2+2\pi i vn+i\pi\tau n^2},$$

where $v_1, \ldots, v_p$ are Riemann's normal integrals of the first kind, the periods of v_r being

$$0, \ldots, 0, 1, 0, \ldots, 0, \quad \tau_{r,1}, \tau_{r,2}, \ldots, \tau_{r,p},$$

vn denotes $v_1n_1+\ldots+v_pn_p$, τn^2 denotes $\Sigma\Sigma\tau_{ij}n_in_j$, $(i, j=1, \ldots, p)$, au^2 denotes $\Sigma\Sigma a_{ij}u_iu_j$, $(i, j=1, 2, \ldots, p)$, wherein $a_{ij}=a_{ji}$, and the $\frac{1}{2}p(p+1)$ quantities $a_{i,j}$ are those occurring in the equation

$$2\sum_{i=1}^{p}\sum_{j=1}^{p} a_{i,j}u_i^{x,a}u_j^{z,c} = \Pi_{z,c}^{x,a} - \int_a^x\int_c^z \frac{dx\,dz}{ys}\cdot\frac{2ys+F(x,z)}{4(x-z)^2},$$

where $\Pi_{z,c}^{x,a}$ is Riemann's normal elementary integral of the third kind, and $F(x, z)$ is any symmetrical integral polynomial in x, z, of degree $p+1$ in each, which satisfies the equations

$$F(z, z)=2f(z), \quad \left[\frac{\partial}{\partial x}F(x, z)\right]_{x=z} = \frac{d}{dz}f(z).$$

The coefficients $c_{i,j}$ are determinable by the fact that

$$\sum_{i=1}^{p}\sum_{j=1}^{p} c_{ij}u_iu_j = \frac{4\left[P(\xi_1)Q(\xi_2)+P(\xi_2)Q(\xi_1)\right]-F(\xi_1, \xi_2)}{4(\xi_1-\xi_2)^2},$$

where the meaning is that after the division on the right-hand has been carried out (as is always possible), we are to put

$$\xi_1^{i-1}=u_i=\xi_2^{i-1}, \quad \xi_1^0=u_1=\xi_2^0. \qquad (i=1, 2, \ldots, p)$$

In particular we may take each of c_{ij} to be zero.

X.—*A fundamental theorem.*

If Q be used, precisely as in No. IX, to denote one of a certain group of 2^p characteristics, then we have the theorem

$$\frac{\vartheta(u+v\,|\,Q)\,\vartheta(u-v\,|\,Q)}{\vartheta^2(u\,|\,Q)\,\vartheta^2(v\,|\,Q)} = \begin{array}{l}\text{rational integral polynomial}\\ \text{in the } p(p+1) \text{ functions}\end{array} \; \frac{\partial^2}{\partial u_i\partial u_j}\log\vartheta(u\,|\,Q),\; \frac{\partial^2}{\partial v_i\partial v_j}\log\vartheta(v\,|\,Q).$$

This follows immediately by combining the results of Nos. VIII and IX.

Two examples may be given; let $\sigma_{12\,\ldots\,k}(u)$ denote the function defined in No. VII; then

(α) for $p=2$, if
$$\wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \partial u_j}\log \sigma_{12}(u),$$
we have
$$-\frac{\sigma_{12}(u+v)\,\sigma_{12}(u-v)}{\sigma_{12}^2(u)\,\sigma_{12}^2(v)} = \wp_{11}(u) - \wp_{11}(v) + (a_1 + a_2 + c_{22})(\wp_{12}(u) - \wp_{12}(v)) \\ + (a_1a_2 - c_{12})(\wp_{22}(u) - \wp_{22}(v)) + \wp_{21}(u)\,\wp_{22}(v) - \wp_{21}(v)\,\wp_{22}(u).$$
In particular, when, with
$$f(x) = a_x^{2p+2} = 4(x^5 + 5A_1x^4 + 10A_2x^3 + 10A_3x^2 + 5A_4x + A_5),$$
we take, as we may,
$$F(x,\,z) = 2a_x^{p+1}a_z^{p+1},$$
we find
$$a_1 + a_2 + c_{22} = -2A_1, \quad a_1a_2 - c_{12} = A_2;$$
and when, with
$$f(x) = 4x^5 + \lambda_4x^4 + \lambda_3x^3 + \lambda_2x^2 + \lambda_1x + \lambda,$$
we take, as we may, with $\lambda_5 = 4,\ \lambda_6 = 0,\ \lambda_7 = 0$,
$$F(x,\,z) = \sum_{i=0}^{p+1} x^iz^i\,[2\lambda_{2i} + \lambda_{2i+1}(x+z)],$$
we find
$$a_1 + a_2 + c_{22} = 0, \quad a_1a_2 - c_{12} = 0.$$

(β). For $p=3$, if
$$\wp_{ij}(u) = -\frac{\partial^2}{\partial u_i\,\partial u_j}\log\sigma_{123}(u),$$
we have
$$-\frac{\sigma_{123}(u+v)\,\sigma_{123}(u-v)}{\sigma_{123}^2(u)\,\sigma_{123}^2(v)}$$
$$= [\wp_{31}(u) - \wp_{31}(v)]^2 - [\wp_{33}(u) - \wp_{33}(v)][\wp_{11}(u) - \wp_{11}(v)] \\ + [\wp_{21}(u) - \wp_{21}(v)][\wp_{23}(u) - \wp_{23}(v)] - [\wp_{22}(u) - \wp_{22}(v)][\wp_{31}(u) - \wp_{31}(v)].$$

Equations of this form, in each of which there occurs *only one function*, and its second logarithmic derivatives, appear to be of importance, not only for the theory of theta functions, but also for the general theory of periodic functions of several independent variables.

Of the functions of u occurring on the right side of these equations, every $p+1$ are connected by an algebraical relation; and they can all be expressed rationally by means of $p+1$ functions having the same periods, and so connected. The discussion of these properties is reserved*—here the expressions are given in the form in which they naturally present themselves in the first instance.

ON THE HYPERELLIPTIC SIGMA FUNCTIONS. DEVELOPMENT.

SECTION I.—*Preliminary lemma.*

A certain factorial function in general.

I. On any Riemann surface, dissected to a p-ply connected surface in the ordinary way, by (a)-cuts or period loops, and (b)-cuts or period loops, there exists one and only one single valued function satisfying the conditions, (i) of being infinite to the first order, with a residue equal to $+1$, at an arbitrary place c, (ii) of vanishing to the first order at an arbitrary place z, (iii) that its value shall be the same at any two opposite points of an (a)-cut, while its value at a point on the left or positive side of a (b)-cut, say (b_r), shall be $e^{2\pi i v_r^{z,c}}$ times its value at the corresponding point of the right side, $v_1^{x,a}, \ldots, v_p^{x,a}$ being Riemann's normal integral of the first kind.

That there cannot be two such functions is clear, since their quotient would be a single-valued function without factors at the (a) and (b) cuts, and without infinities, and having the value unity at the place c.

While, on the other hand, if K be a properly chosen quantity, independent of x, and $\Pi_{z,c}^{x,k}$ be Riemann's normal elementary integral of the third kind, with z and c as infinities, chosen so as to vanish at the place k, the function is clearly given by $Ke^{\Pi_{z,c}^{x,k}}$.

There are reasons however why this form is not convenient; we therefore introduce another. Let

$$\Theta(u) = \Sigma e^{2\pi i n u + i\pi\tau n^2},$$

where nu denotes $n_1u_1 + \ldots + n_pu_p$, τn^2 denotes $\Sigma\Sigma\tau_{ij}n_in_j$, and the summation is in regard to each of $n_1, \ldots n_p$ over all integers from $-\infty$ to $+\infty$; further,

* [*Added, September*, 1898. The reader may refer to a note in the Proceedings of the Cambridge Philosophical Society, May Term, 1898 (vol. IX, Part IX)].

let $\frac{1}{2}\Omega$ be any *odd* half-period, that is, a set of p simultaneous half-periods, of which one element is given by

$$\tfrac{1}{2}(\Omega)_i = s_i + \tau_{i,1}s_1' + \ldots + \tau_{i,p}s_p', \qquad (i = 1, 2, \ldots, p)$$

then, assuming as known the fact that the function of x expressed by $\Theta(v^{x,z} + \frac{1}{2}\Omega)$ has p zeros of the first order, one of them at the place z and the others independent of z, and has no infinities, the factorial function in question is clearly given by

$$f(x;\, z,\, c) = \frac{\Theta(v^{x,z} + \frac{1}{2}\Omega)}{\Theta(v^{x,c} + \frac{1}{2}\Omega)} \frac{\sum_{i=1}^{p}(Dv_i)_c\Theta_i'(\frac{1}{2}\Omega)}{\Theta(v^{c,z} + \frac{1}{2}\Omega)},$$

wherein $\sum_{i=1}^{p}(Dv_i)_c\Theta_i'(\frac{1}{2}\Omega)$ represents the value, at the place c, of the differential coefficient of the function $\Theta(v^{x,c} + \frac{1}{2}\Omega)$.

II. For the function $f(x;\, z,\, c)$ we have the property

$$f(z';\, z,\, c) = -f(z;\, z',\, c),$$

namely, the value at any place z', of the function defined in (I), with c as pole and z as zero, is the negative of the value, at the place z, of the corresponding function with the same pole c but a different zero z'.

This is immediately obvious from the representation given, bearing in mind the equation

$$\Theta(v^{z,x} + \tfrac{1}{2}\Omega) = -e^{-2\pi i s' v^{x,z}}\Theta(v^{x,z} + \tfrac{1}{2}\Omega),$$

where $s'v^{x,z}$ is, for abbreviation, put for $s_1'v_1^{x,z} + \ldots + s_p'v_p^{x,z}$. This equation expresses a fundamental property of the function Θ.

III. The proof of the property in (II) can be given in a more elementary way in terms of integrals of the third kind only. For that purpose we require a result contained in the following proposition:

If $P_{z,c}^{x,k}$ represent any elementary integral of the third kind, with infinities at z and c, and chosen so as to vanish at k, and if x, x_1, x_2, x_3 be any places whatever, we have

$$P_{x_2,x_3}^{x,x_1} + P_{x_3,x_1}^{x,x_2} + P_{x_1,x_2}^{x,x_3} = \text{odd integral multiple of } \pi i.$$

To prove this we notice first that it is sufficient to prove it for the case where the integral of the third kind is taken to be Riemann's normal elementary

integral of the third kind; the more general statement immediately follows from this. Now if (x, z) be used for shortness to denote the function $\Theta(v^{x, z} + \frac{1}{2}\Omega)$, it can be easily proved, by comparing the zeros and poles of the two sides of the equation and the factors at the period-loops, that the following equation holds:

$$\Pi_{z, c}^{x, k} = \frac{[x, z]\,[k, c]}{[x, c]\,[k, z]};$$

hence we have

$$e^{\Pi_{x_2, x_3}^{x, x_1} + \Pi_{x_3, x_1}^{x, x_2} + \Pi_{x_1, x_2}^{x, x_3}} = \frac{[x, x_2][x_1, x_3]}{[x, x_3][x_1, x_2]} \cdot \frac{[x, x_3][x_2, x_1]}{[x, x_1][x_2, x_3]} \; \frac{[x, x_1][x_3, x_2]}{[x, x_2][x_3, x_1]},$$

and, in virtue of the equation $[x, z] = -e^{2\pi i s' v^{x, z}}[z, x]$, already quoted, the right side is equal to -1.

Cor. 1. Putting $x = x_1$ and, for greater clearness, replacing x_1, x_2, x_3 respectively by z, a, b, we have the result

$$limit\left(e^{\Pi_{z, a}^{x, b} - \Pi_{z, b}^{x, a}}\right)_{x=z} = -1.$$

This includes the general form of the proposition.

Cor. 2. In order to use the result of (III) to prove the result of (II), notice that, ξ being an arbitrary place,

$$\frac{f(x;\, z, c)}{f(\xi;\, z, c)} = e^{\Pi_{z, c}^{x, \xi}}, \quad \frac{f(x;\, z', c)}{f(\xi;\, z', c)} = e^{\Pi_{z', c}^{x, \xi}},$$

and therefore

$$\frac{f(z';\, z, c)}{f(z;\, z', c)} = \frac{f(\xi;\, z, c)}{f(\xi;\, z', c)}\, e^{\Pi_{z, c}^{z', \xi} - \Pi_{z', c}^{z, \xi}};$$

while also

$$\frac{f(x;\, z, c)}{f(x;\, z', c)} = e^{\Pi_{z, z'}^{x, c}};$$

thus

$$\frac{f(z';\, z, c)}{f(z;\, z', c)} = e^{\Pi_{z, z'}^{\xi, c} + \Pi_{z', c}^{\xi, z} + \Pi_{c, z}^{\xi, z'}} = -1,$$

which is the result we desired.

This proof of the result in (II) is to be preferred to the former one, because, although the theorem for integrals of the third kind upon which it depends has here been deduced from the properties of theta functions, that theorem is, I think, clearly capable of a more elementary proof by contour integration.

Cor. 3. It is easy to see that

$$f(x;\, z,\, c)\, e^{-\Pi_{z,\,c}^{x,\,z'}} + f(x';\, z',\, c)\, e^{-\Pi_{z',\,c}^{x',\,z}} = 0\,.$$

In case x, x' coincide respectively with z', z, this becomes the proposition in (II).

The results in this Section I are of a preliminary character, and in what follows will only be used for the hyperelliptic case.

SECTION II.

Of methods of dissecting the hyperelliptic Riemann Surface.

The fundamental algebraic equation is taken to be

$$y^2 = 4\,(x-c_1)(x-a_1)\,\ldots\,(x-c_p)(x-a_p)(x-c)\,,\; = f(x)\,,$$

wherein $c_1, a_1, \ldots, c_p, a_p, c$ are given (complex or real) quantities. Over the x plane we suppose a two-sheeted Riemann surface to be constructed, with $p+1$ cross lines, between the places $(c_1,\, a_1)$, $(c_2,\, a_2)$, $\ldots$, $(c_p,\, a_p)$, $(c,\, \infty)$. We suppose the branch values and the branch places $c_1, a_1, \ldots$ always to be taken in the order

$$c_1,\, a_1,\, c_2,\, a_2,\, \ldots,\, c_r,\, a_r,\, \ldots,\, c_p,\, a_p,\, c,\, \infty,$$

and shall often denote them respectively by

$$b_1,\, b_2,\, b_3,\, b_4,\, \ldots,\, b_{2r-1},\, b_{2r},\, \ldots,\, b_{2p-1},\, b_{2p},\, b_{2p+1},\, a,$$

using throughout the symbol a for the branch place at infinity. By the *standard case* we mean that in which $c_1, a_1, \ldots, a_p, c$ are real and in ascending order of magnitude; by the difference $b_i \sim b_j$ we mean that in which the b of less suffix is subtracted from that of greater suffix, namely, $i > j$; thus in the standard case $b_i \sim b_j$ will be positive as well as real. We suppose the Riemann surface to be changed into a p-ply connected surface in the ordinary way, by means of p period-loop-pairs, the sides of each of these pairs constituting a closed curve, and speak of the resulting surface as the dissected Riemann surface, using sometimes by analogy the word cuts in place of period loops.

Let $u_1^{x,\,a}, \ldots, u_p^{x,\,a}$ be any p linearly independent integrals of the first kind, single-valued upon the dissected surface; denote the periods of $u_r^{x,\,a}$ at the first and second of the i^{th} pair of period-loops, namely, at the loops (a_i), (b_i),

respectively by $2\omega_{r,i}$, $2\omega'_{r,i}$. Then if b denote any one of the $(2p+1)$ finite branch places, we have

$$u_r^{a,b} = \beta_1\omega_{r,1} + \ldots + \beta_p\omega_{r,p} + \beta'_1\omega'_{r,1} + \ldots + \beta'_p\omega'_{r,p}, \qquad (r = 1, \ldots, p)$$

where $\beta_1, \ldots, \beta'_p$ are integers, which are immediately obvious by inspection in case of any specified dissection of the surface; these equations we denote by

$$u^{a,b} = \tfrac{1}{2}\binom{\beta'}{\beta};$$

upon the properties of the $(2p+1)$ half-integer characteristics thus arising depends the definiteness of the determination of the signs of certain square roots arising here.

If Q, K be any two half-integer characteristics given by

$$Q = \tfrac{1}{2}\binom{q'}{q} = \tfrac{1}{2}\binom{q'_1, \ldots, q'_p}{q_1, \ldots, q_p}, \quad K = \tfrac{1}{2}\binom{k'}{k} = \tfrac{1}{2}\binom{k'_1, \ldots, k'_p}{k_1, \ldots, k_p},$$

we use the abbreviations

$$\left(\frac{Q}{K}\right) = e^{\pi i q'k} = exp.\ \pi i \sum_{r=1}^{p} q'_r k_r, \quad |Q| = qq' = \sum_{r=1}^{p} q_r q'_r,$$

$$|Q, K| = qk' - q'k = \sum_{r=1}^{p}(q_r k'_r - q'_r k_r),$$

so that

$$e^{\pi i |Q, K|} = \left(\frac{Q}{K}\right)\left(\frac{K}{Q}\right), \quad e^{\pi i |Q|} = \left(\frac{Q}{Q}\right);$$

further, we denote the characteristic associated with the half-period u^{a,b_i} by B_i; for our purpose the values of the $2p(2p+1)$ quantities

$$\left(\frac{B_i}{B_j}\right), \quad |B_i, B_j|$$

will be found to be of importance.

It is convenient to choose the dissection of the Riemann surface so that, provided $i > j$, the values of these quantities shall be independent of i and j; we give therefore below examples of methods of dissection whereby

(I) when $i > j$,

$$|B_i, B_j| = 1, \quad \left(\frac{B_i}{B_j}\right) = 1, \text{ and therefore } |B_j, B_i| = -1, \quad \left(\frac{B_j}{B_i}\right) = -1,$$

(II) when $i > j$,

$$|B_i, B_j| = -1, \quad \left(\frac{B_i}{B_j}\right) = 1, \text{ and therefore } |B_j, B_i| = 1, \quad \left(\frac{B_j}{B_i}\right) = -1,$$

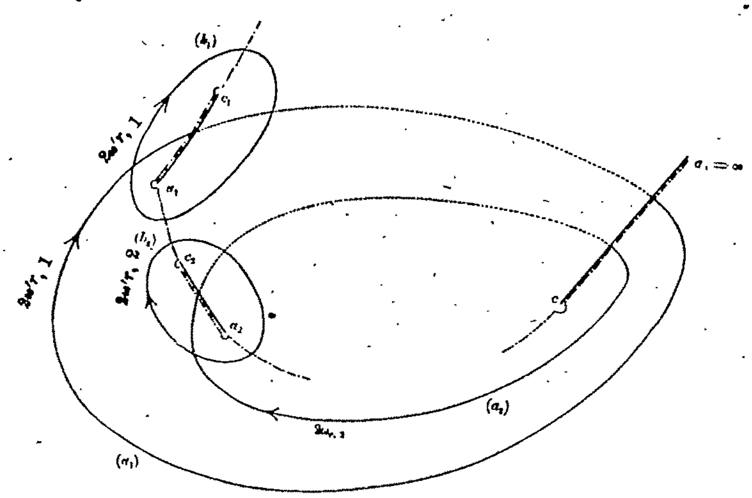
(III) when $i > j$,

$$|B_i, B_j| = \ \ 1, \quad \binom{B_i}{B_j} = -1, \text{ and therefore } |B_j, B_i| = -1, \quad \binom{B_j}{B_i} = \ \ 1,$$

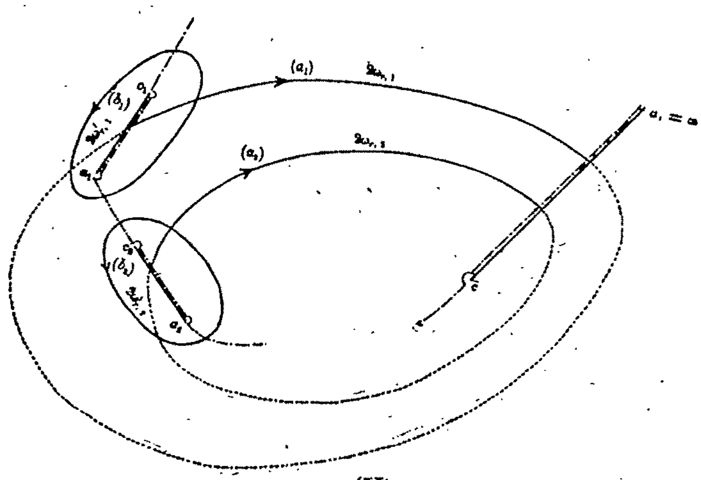
(IV) when $i > j$,

$$|B_i, B_j| = -1, \quad \binom{B_i}{B_j} = -1, \text{ and therefore } |B_j, B_i| = \ \ 1, \quad \binom{B_j}{B_i} = \ \ 1,$$

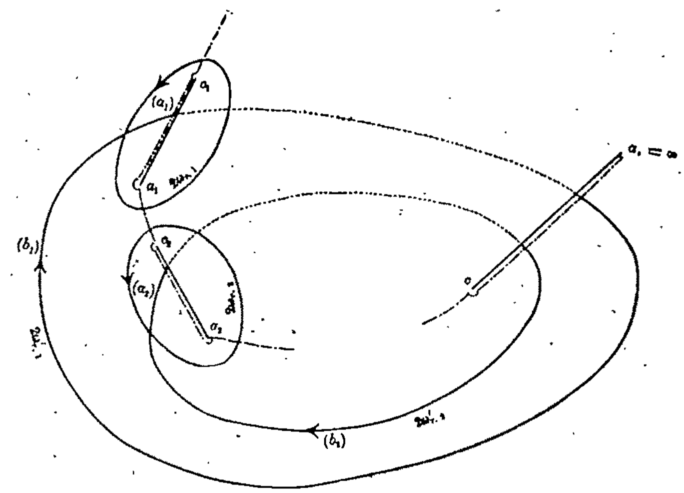
namely, such dissections are represented diagrammatically by the four following figures respectively:



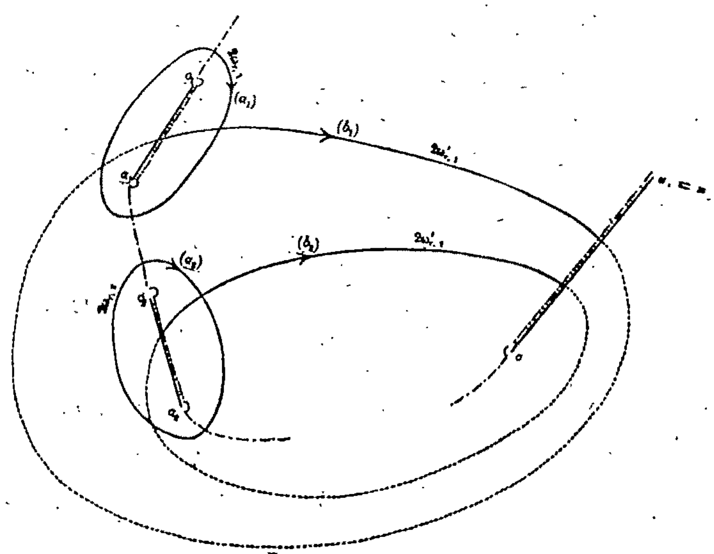
(I)



(II)



(III)



(IV)

We find for these dissections respectively,

$$\text{(I)} \qquad u^{a,\,c_r} = \tfrac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}^{r-1}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}^{p-r},$$

$$u^{a,\,a_r} = \tfrac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}^{r-1}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}^{p-r}, \quad u^{a,\,c} = \tfrac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}^{p},$$

(II) $u^{a,\,c_r} = \tfrac{1}{2}\binom{0}{1}^{r-1}\binom{-1}{0}\binom{0}{0}^{p-r}$,

$$u^{a,\,a_r} = \tfrac{1}{2}\binom{0}{1}^{r-1}\binom{-1}{1}\binom{0}{0}^{p-r}, \quad u^{a,\,c} = \tfrac{1}{2}\binom{0}{1}^{p},$$

(III) $u^{a,\,c_r} = \tfrac{1}{2}\binom{1}{0}^{r-1}\binom{0}{-1}\binom{0}{0}^{p-r}$,

$$u^{a,\,a_r} = \tfrac{1}{2}\binom{1}{0}^{r-1}\binom{1}{-1}\binom{0}{0}^{p-r}, \quad u^{a,\,c} = \tfrac{1}{2}\binom{1}{0}^{p},$$

(IV) $u^{a,\,c_r} = \tfrac{1}{2}\binom{1}{0}^{r-1}\binom{0}{1}\binom{0}{0}^{p-r}$,

$$u^{a,\,a_r} = \tfrac{1}{2}\binom{1}{0}^{r-1}\binom{1}{1}\binom{0}{0}^{p-r}, \quad u^{a,\,c} = \tfrac{1}{2}\binom{1}{0}^{p};$$

from these the facts stated above as to the values of the quantities $|B_i,\, B_j|$, $\binom{B_i}{B_j}$ can be immediately verified.

If we denote the periods of an integral $u_r^{x,\,a}$, for the case (I), at the i^{th} period loops of the first and second kind respectively, by $2\omega_{r,\,i}$, $2\omega'_{r,\,i}$, and denote the corresponding periods for the dissections (II), (III), (IV) respectively by

$$[2\omega_{r,\,i}]_2,\ [2\omega'_{r,\,i}]_2;\ [2\omega_{r,\,i}]_3,\ [2\omega'_{r,\,i}]_3;\ [2\omega_{r,\,i}]_4,\ [2\omega'_{r,\,i}]_4,$$

we immediately find, supposing of course that the values of y are the same, at any the same place of the surface, in all the four cases, that

$$\begin{matrix} [\omega_{r,\,i}]_2 = -\omega_{r,\,i}, & [\omega_{r,\,i}]_3 = -\omega'_{r,\,i}, & [\omega_{r,\,i}]_4 = \omega'_{r,\,i}, \\ [\omega'_{r,\,i}]_2 = -\omega'_{r,\,i}, & [\omega'_{r,\,i}]_3 = \omega_{r,\,i}, & [\omega'_{r,\,i}]_4 = -\omega_{r,\,i}, \end{matrix} \quad \begin{pmatrix} r = 1,\, 2,\, \ldots,\, p \\ i = 1,\, 2,\, \ldots,\, p \end{pmatrix}$$

thus the matrices associated with the linear transformation (B. 531) from case (I) to the cases (II), (III), (IV) respectively, are those denoted by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is to be observed that from each of the four dissections (I), (II), (III), (IV) we can immediately derive another by changing the direction of every (a) loop and every (b) loop. Thus, from (1) we can derive a dissection in which the periods are exactly equal to those arising in (II)—in which however the characteristics associated with the half-periods $u^{a,\,c_r}$, $u^{a,\,a_r}$, $u^{a,\,c}$ will have all their elements negative. The quantities $|B_i,\, B_j|$, $\binom{B_i}{B_j}$ are, however, unaltered by the alteration of direction of every (a) loop and every (b) loop.

SECTION III.

Of the construction of the functions Θ.

Denoting the characteristic associated with a half-period $u^{a,b}$ by $\frac{1}{2}\binom{q'}{q}$ and the matrices of the periods $2\omega_{r,i}$, $2\omega'_{r,i}$ respectively by 2ω, $2\omega'$, we obtain the normal integrals v and the symmetrical matrix τ by the equations

$$u = 2\omega v, \quad \omega' = \omega\tau;$$

then the function $\Theta(u|u^{a,b})$ or $\Theta(u|B)$ is given by

$$\sum_n e^{2\pi i v(n+\frac{1}{2}q') + i\pi\tau(n+\frac{1}{2}q')^2 + i\pi q(n+\frac{1}{2}q')},$$

where $v(n+\frac{1}{2}q')$ denotes $\sum_{r=1}^{p} v_r(n_r+\frac{1}{2}q'_r)$, $\tau(n+\frac{1}{2}q')^2$ denotes $\sum_{r=1}^{p}\sum_{s=1}^{p}\tau_{r,s}(n_r+\frac{1}{2}q'_r)(n_s+\frac{1}{2}q'_s)$, $q(n+\frac{1}{2}q')$ denotes $\sum_{r=1}^{p} q_r(n_r+\frac{1}{2}q'_r)$, and the summation $\sum_n$ is in regard to each of $n_1, \ldots, n_p$ over all positive and negative integers. It follows that, for the same values of the arguments v, the function is unaffected by the addition of integers to the half-integers $\frac{1}{2}q'_1, \ldots, \frac{1}{2}q'_p$ occurring in the upper line of the characteristic $\frac{1}{2}\binom{q'}{q}$; thus the function $\Theta(u|u^{a,b})$ as defined by the dissection (II) differs from the function $\Theta(u|u^{a,b})$ as defined by the dissection (I) only in the sign of the arguments v; that is, the even functions in (II) are equal to those in (I), but the odd functions have a different sign. Similarly the functions constructed for the dissection (IV) are derived from those for the dissection (I) by a formula given B. 558; the ratio of any function (I) to the corresponding function (IV) consists of a factor independent of the characteristic, multiplied by ε, where ε is 1 in the case of the even functions, and i in the case of the odd functions.

It is clear that the theta functions arising for any one of the four dissections will be unaltered by changing the direction of every (a) loop and every (b) loop; for this change alters the sign of the characteristic and also the sign of the normal integrals.

Further, if $B_r = \frac{1}{2}\binom{q'}{q}$, $B_s = \frac{1}{2}\binom{k'}{k}$ be the two characteristics associated

respectively with the half-periods u^{a_r, b_r}, u^{a_s, b_s}, and

$$q'_r + k'_r = 2M'_r + \lambda'_r, \quad q_r + k_r = 2M_r + \lambda_r, \quad (r = 1, 2, \ldots, p)$$

where λ_r, λ'_r are integers each $= 0$ or 1, and M_r, M'_r are integers, then, by the *reduced* characteristic $B_r B_s$ is meant the characteristic $\frac{1}{2}\begin{pmatrix}\lambda'\\ \lambda\end{pmatrix}$. It is desirable to compare the dissections (I), (II), (III), (IV) in regard to the values of the integers (M, M'); we have the equation

$$\Theta\left[u \left| \begin{pmatrix}M'\\ M\end{pmatrix} + \tfrac{1}{2}\begin{pmatrix}\lambda'\\ \lambda\end{pmatrix}\right.\right] = e^{\pi i M\lambda'}\,\Theta\left(u \left| \tfrac{1}{2}\begin{pmatrix}\lambda'\\ \lambda\end{pmatrix}\right.\right);$$

hence, denoting the factor $e^{\pi i M\lambda'}$ as *the reduction factor for the places* b_r, b_s, we find that for the dissections

(I) and (II), the reduction factor is -1 for all cases in which the less of the two values b_r, b_s—that is, the one of these occurring first in the ascending order of the $2p+1$ finite branch places—is one of $a_1, \ldots, a_p$, and is otherwise $+1$.

(III) and (IV), the reduction factor is -1 when the places b_r, b_s are such a pair as c_r and a_r, and is otherwise $+1$.

In order to avoid the consideration of the reduction factor we shall frequently in what follows use $\Theta(u \,|\, B_r B_s)$ to denote the function in which the characteristic is *not* reduced, namely the function $\Theta\left[u \left| \tfrac{1}{2}\begin{pmatrix}q'+k'\\ q+k\end{pmatrix}\right.\right]$, and for this purpose we shall reverse the ordinary rule: *Unless the contrary is stated it will be intended in the function* $\Theta(u \,|\, B_r B_s)$ *that the characteristic is unreduced.*

SECTION IV.

Of the fundamental radical functions on the Riemann-surface.

On the dissected surface, if b denote any one of the finite branch places, either one of the square roots of $x - b$ is single-valued; this follows from the fact that $x - b$ vanishes to the second order at b and is infinite to the second order at infinity; to fix the value of this square root it is only necessary to specify the sign for some one value of x; we suppose the sign chosen for $x = \infty$, and that this sign is the same for each of the $2p+1$ functions in which b is in turn every one of the finite branch places; supposing that the values of y have

been allocated to the places of the surface before dissection, and noticing that the sign of a product of $2p+1$ factors is altered if the sign of every factor is altered, we agree that the signs of the functions considered, which functions are henceforth to be denoted simply by symbols $\sqrt{x-b}$, shall be such that for a point in the lower sheet at infinity, the ratio

$$\tfrac{1}{2} y \Big/ \prod_{r=1}^{2p+1} \sqrt{x-b_r}$$

shall be equal to $+1$; since this ratio is a single-valued continuous function whose square is $+1$, it follows that, for every place of the surface,

$$y = 2 \prod_{r=1}^{2p+1} \sqrt{x-b_r};$$

that is, $$y = 2\sqrt{x-c_1}\sqrt{x-a_1} \ldots \sqrt{x-a_p}\sqrt{x-c}.$$

Further, for the sake of definiteness, we suppose the sign of the infinitesimal at infinity, which is such that $x = t^{-2}$, so chosen that at infinity, for each of the functions $\sqrt{x-b}$,

$$\lim_{t=0} (t\sqrt{x-b}) = +1.$$

We may further suppose for real infinite x, and in the lower sheet of the surface, that t is a real *positive* quantity. Thus in the standard case, for all real values of x greater than b, $\sqrt{x-b}$ would be positive as well as real.

Suppose now that, in the abbreviated notation before explained,

$$u^{a,b} = \tfrac{1}{2}\binom{\beta'}{\beta},$$

and let $\frac{1}{2}\Omega$ denote an odd half-period—that is, a set of p elements each of the form

$$\tfrac{1}{2}(s_i + \tau_{i,1}s_1' + \ldots + \tau_{i,p}s_p'), \qquad (i = 1, 2, \ldots, p)$$

wherein $s_1, \ldots, s_p, s_1', \ldots, s_p'$ are integers; then, on the dissected surface, the function

$$\sqrt{x-b}\,\frac{\Theta(v^{x,a}+\frac{1}{2}\Omega)}{\Theta(v^{x,b}+\frac{1}{2}\Omega)}\;\frac{\Theta(v^{a,b}+\frac{1}{2}\Omega)}{\sum_{i=1}^{p}(Dv_i)_a\,\Theta_i'(\frac{1}{2}\Omega)}\,e^{-\pi i\beta' v^{x,a}},$$

wherein the second factor of the denominator denotes the value at infinity of the differential coefficient in regard to the infinitesimal of the function $\Theta(v^{x,\,a}+\tfrac{1}{2}\Omega)$, is single-valued and continuous, and has no zeros or infinities; at the r^{th} period loops respectively of the first and second kind the square of the function has the factors

$$(e^{-\pi i\beta_r'})^2, \quad (e^{2\pi i v_r^{b,\,a}-\pi i\tau_r\beta'})^2=(e^{\pi i\beta_r})^2,$$

each of which is equal to unity; thus the square of the function, being equal to $+1$ at the place infinity of the surface, is everywhere equal to $+1$. Thus the function itself, which is also equal to $+1$ at infinity, is also everywhere equal to $+1$. Hence, in the notation employed in Section I, we have

$$\sqrt{x-b}=e^{\pi i\beta' v^{x,\,a}}f(x;\,b,\,a),$$

and this result is independent of the method of dissection of the surface.

It follows that the factors of $\sqrt{x-b}$ at the r^{th} period loops respectively of the first and second kind are

$$e^{\pi i\beta_r'}, \quad e^{\pi i\tau_r\beta'+2\pi i v_r^{b,\,a}}=e^{\pi i\beta_r};$$

this fact we shall denote by

$$\sqrt{x-b}\propto\binom{\beta}{\beta'},$$

and it will be found that this relation may be supposed to express another fact also. The factors of $\sqrt{x-b}$ at the period loops are immediately obvious geometrically by considering the number of revolutions about the branch-place b involved in taking the function round the period loops. The analytical proof has been preferred because it brings into greater prominence the fact that the result obtained is the same for any method of dissection.

Let now d be another finite branch-place such that

$$u^{a,\,d}=\tfrac{1}{2}\binom{\delta'}{\delta};$$

let the signs $\sqrt{d-b}$, $\sqrt{b-d}$ denote respectively the value of the function $\sqrt{x-b}$ at d, and the value of the function $\sqrt{x-d}$ at b; then it follows from Section I that

$$\sqrt{d-b}\,e^{-\pi i\beta' v^{d,\,a}}=-\sqrt{b-d}\,e^{-\pi i\delta' v^{b,\,a}},$$

and therefore

$$\frac{\sqrt{d-b}}{\sqrt{b-d}} = -e^{\frac{1}{2}\pi i[\delta'(\beta+\tau\beta')-\beta'(\delta+\tau\delta')]} = -e^{-\frac{1}{2}\pi i|D,\,B|},$$

where $B = \frac{1}{2}\binom{\beta'}{\beta}$, $D = \frac{1}{2}\binom{\delta'}{\delta}$; by squaring it follows that for *every method of dissection*

$$|B,\,D| \equiv 1, \qquad (\text{mod. } 2)$$

and therefore that, also for any method of dissection,

$$\frac{\sqrt{d-b}}{\sqrt{b-d}} = e^{\frac{1}{2}\pi i|D,\,B|}.$$

Hence, *in all methods of dissection for which, when* $i > j$, $|B_i,\,B_j| \equiv 1$ (*mod.* 4), *we have* $\sqrt{b_j - b_i} = -i\sqrt{b_i - b_j}$; *in all methods of dissection for which, when* $i > j$, $|B_i,\,B_j| \equiv -1$ (*mod.* 4), *we have* $\sqrt{b_j - b_i} = +i\sqrt{b_i - b_j}$.

With more precise conventions we may use the result somewhat differently, namely, if for two half-integer characteristics $Q = \frac{1}{2}\binom{q'}{q}$, $K = \frac{1}{2}\binom{k'}{k}$, the symbol $\sqrt{\binom{Q}{K}}$ denote $e^{\frac{1}{2}\pi i q'k}$, we have $\sqrt{\binom{D}{B}}\sqrt{d-b} = \sqrt{\binom{B}{D}}\sqrt{b-d}$.

In what follows *any one of the* $p(2p+1)$ *signs* $\sqrt{b_i - b_j}$ *will be used to mean the value of the function* $\sqrt{x - b_j}$ *when* x *is at* b_i; and, in accordance with the established practice in the elliptic case (Schwarz, "Formeln u. Lehrsätze," p. 24; Halphen, "Fonct. Ellipt.," I, p. 192), we shall take for the normal method of dissection one in which, for $i > j$, $|B_i,\,B_j| \equiv 1$ (mod. 4), as in the methods (I), (III) above (pp. 320 and 321); such a method may be described as a negative method, those in which $|B_i,\,B_j| \equiv -1$ (mod. 4) being described as positive methods of dissection.

It is easy to see geometrically, in the figures (I) and (III), in what way the negative sign arises; let a continuous line be drawn in the lower sheet of the surface from $+\infty$ to $-\infty$—indicated by the dotted line in the figures—so as to pass near all the cross lines of the surface; in reaching a branch-place this line may be supposed to describe a semicircle in the *clockwise* direction; thus the description of the semicircle near a branch-place b gives for the function $\sqrt{x-b}$ a factor $-i$. In the dissections (II), (IV) the corresponding semicircles are

described in the counterclockwise direction, and give a factor $+i$ for $\sqrt{x-b}$ as the describing point passes round the branch-place b.

The question may arise whether the method of proof just given, depending on the theta functions, may not be vitiated in the hyperelliptic case owing to the occurrence of even functions which vanish for zero values of the arguments; it is for this reason we have, in Section I, given an alternative proof, depending on the fact that if $P^{x,\,a}_{x_1,\,x_2}$ denote an elementary integral of the third kind, with its infinities at x_1, x_2, then

$$P^{x,\,x_3}_{x_1,\,x_2}+P^{x,\,x_1}_{x_2,\,x_3}+P^{x,\,x_2}_{x_3,\,x_1}=\text{odd integral multiple of }\pi i;$$

though we have deduced this property by means of the theta functions, it appears clear that it is capable of an elementary proof.

We shall for convenience introduce, beside the function denoted by $\sqrt{x-b}$, a function denoted by $\sqrt{b-x}$, and defined, for every one of the finite branch-places b, by the equation, holding for every value of x,

$$\sqrt{b-x}=+i\sqrt{x-b},$$

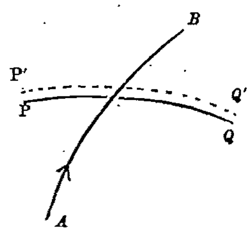
but some care must be exercised in regard to it; the value

$$(\sqrt{b_i-x})_{x=b_j},\ =(i\sqrt{x-b_i})_{x=b_j}=i\sqrt{b_j-b_i}=,\ ie^{\frac{1}{2}\pi i\,|B_j,\,B_i|}\sqrt{b_i-b_j},$$

is equal to $\sqrt{b_i-b_j}$ only when $|B_i,\ B_j|\equiv 1$ (mod. 4); that is, if for $i>j$, $|B_i,\ B_j|\equiv 1$ (mod. 4), *$\sqrt{b_i-b_j}$ can be interpreted as the value of $\sqrt{b_i-x}$ when x is at b_j only if, in the ascending order of the branch-places, b_i has a higher place than b_j.* It is for this reason we introduce, and shall retain, the convention of using $\sqrt{b_i-b_j}$ only as the value of $\sqrt{x-b_j}$ when x is at b_i.

The function $\sqrt{x-b}$, when x is near to b, has opposite signs in the conjugate places of the Riemann surface—as is obvious by considering that we pass from one of these places to the other by a path going once round the branch-place b. But for values of x which are near to any other finite branch-place, the conjugate values of $\sqrt{x-b}$ are not different, since the function would otherwise vanish at this branch-place. The question then arises of specifying the range of values of x for which the conjugate values of the function $\sqrt{x-b}$ are the same; knowing the factors of $\sqrt{x-b}$ at the period loops, it is easy to do this.

Let AB be a portion of a period loop, P, Q being two places on opposite



sides of this loop, in the same sheet as the loop, P being on the left side of the loop; let P', Q' be their conjugate places; let ϵ be the factor of $\sqrt{x-b}$ at this period loop, $\zeta_Q (= \pm 1)$ be the ratio of the values of $\sqrt{x-b}$ at Q and Q', and ζ_P the ratio of the values at P and P', then we clearly have $\zeta_P = \epsilon\zeta_Q$. By this remark it is easy to prove the following rule: Let the period loops be projected upon the plane below the Riemann surface; the projections will form a network of closed curves which, for the moment, we may distinguish, from the period loops themselves, as *barriers;* let the conjugate places of the Riemann surface which project, on to this plane, into a point lying within the barrier arising from any period loop, be called the places *within* this period loop; then, *if in the relation*

$$\sqrt{x-b} \propto \binom{\beta}{\beta'},$$

formerly employed, there occurs a unity in the r^{th} place of the upper line of the symbol on the right, and a zero in the r^{th} place of the lower line, it means that the values of $\sqrt{x-b}$ are the same for the conjugate places within the loop (b_r); if there occurs a unity in the r^{th} place of the lower line of the symbol on the right, and a zero in the r^{th} place of the upper line, it means that the values of $\sqrt{x-b}$ are the same for the conjugate places within the loop (a_r); if there occurs a unity in the r^{th} place of both the upper and the lower lines of the symbol on the right, it means that the values of $\sqrt{x-b}$ are the same within the loop (a_r), and also within the loop (b_r), except for that region which is within both (a_r) and (b_r), where the conjugate values of $\sqrt{x-b}$ are different. For all regions not specified in this enumeration the conjugate values are different.

For instance, if we take the dissections (I) or (II) and denote the barriers which are the projections of the period loops (a_r) and (b_r) respectively by (a_r') and (b_r'),

the function $\sqrt{x - c_r}$ has its conjugate values equal for all values of x within the non-intersecting closed curves formed by the barriers (a'_r), (b'_1), ..., (b'_{r-1}); for other values of x its conjugate values are opposite;

the function $\sqrt{x - a_r}$ has its conjugate values equal within the barriers (a'_r), (b'_1), ..., (b'_r), save in the region common to the closed curves (a'_r) and (b'_r), where they are opposite; for values of x not enclosed by any of these barriers (a'_r), (b'_1), ..., (b'_r), the values of $\sqrt{x - a_r}$ are opposite;

the function $\sqrt{x - c}$ has its conjugate values opposite except within the closed curves formed by the barriers (b'_1), ..., (b'_p).

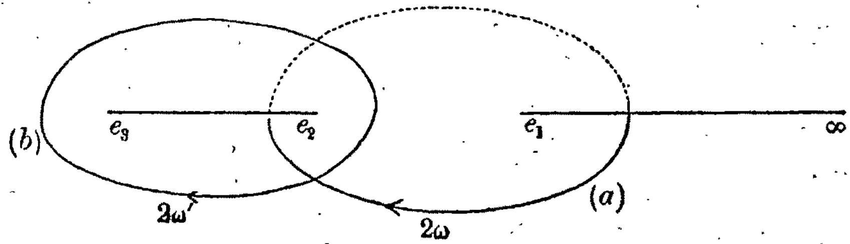
These facts follow from the values of the half-periods u^{a, c_r}, u^{a, a_r}, $u^{a, c}$ given on pp. 322 and 323. The corresponding results for the dissections (III) and (IV) can be obtained in the same way, or are geometrically obvious from these.

To every function $\sqrt{x - b}$ there are then certain characteristic period loops, namely, those at which its factor is -1. On the Riemann surface the function $\sqrt{x - b}$ does not take every value, for it can take the values $\pm B$ only when $x = b + B^2$, and it may happen that for this value of x the conjugate values of $\sqrt{x - b}$ are the same. If, however, we take two precisely similar and equal exemplars of the Riemann surface and make opposite conventions for the sign of $\sqrt{x - b}$ on these two surfaces, so that at any two corresponding points, one on each surface, the values of $\sqrt{x - b}$ are equal and opposite, and join these two exemplars into one surface by converting the characteristic period loops of the function $\sqrt{x - b}$ into cross lines between the two surfaces, we shall obtain a surface of four sheets whereon $\sqrt{x - b}$ is continuous and single-valued, and takes every value twice. The behavior of $\sqrt{x - b}$ on either of the exemplars is such that if the values of $\sqrt{x - b}$ arising thereon be mapped upon a plane, the plane will be covered once with the exception of k holes, corresponding to the points of which no values of $\sqrt{x - b}$ arise—k being the number of characteristic loops belonging to $\sqrt{x - b}$.

We have proved that the product of the $2p + 1$ functions $\sqrt{x - b}$ is $\frac{1}{2} y$. It follows that for every value of x there is an even number, or none, of these functions, of which the conjugate values are the same.

APPENDIX I TO SECTION IV. *Application to the Elliptic case.*

Replacing the letters c_1, a_1, c which have denoted the branch places, respectively by e_3, e_2, e_1, and taking a dissection of the type (I), we have



$$u^{\infty, e_1} = \omega = \tfrac{1}{2}\binom{0}{1}, \quad u^{\infty, e_2} = \omega + \omega' = \tfrac{1}{2}\binom{1}{1}, \quad u^{\infty, e_3} = \omega' = \tfrac{1}{2}\binom{1}{0};$$

also

$\sqrt{x - e_1}$, $\propto \binom{1}{0}$, has factor -1 at loop (b); its conjugate values inside (b) are equal,

$\sqrt{x - e_2}$, $\propto \binom{1}{1}$, has factor -1 at loops (a) and (b); its conjugate values are the same inside (a) and inside (b) except for values of x inside both (a) and (b), where they are opposite,

$\sqrt{x - e_3}$, $\propto \binom{0}{1}$, has factor -1 at loop (a); its conjugate values inside (a) are the same.

The values of $u^{x, \infty}$, $= \int_\infty^x \frac{dx}{y}$, in the two sheets of the surface, (i) outside both the loops (a) and (b) are, for conjugate places, u and $-u$; (ii) inside loop (a), at conjugate places, are u and $-u - 2\omega$; (iii) inside both the loops (a) and (b), at conjugate places of the surface, are u and $-u - 2\omega - 2\omega'$; (iv) inside the loop (b), at conjugate places, are $-u - 2\omega'$; hence putting

$$\chi_1(u) = \frac{\sigma_1(u)}{\sigma(u)} = \sqrt{x - e_1}, \quad \chi_2(u) = \sqrt{x - e_2}, \quad \chi_3(u) = \sqrt{x - e_3},$$

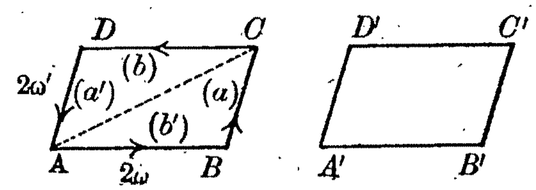
the statements in regard to the three radical functions are easily seen to be summed up in the statements that $\chi_1(u)$, $\chi_2(u)$, $\chi_3(u)$ are odd functions with the respective periods, (i) for $\chi_1(u)$, 2ω, $4\omega'$; (ii) for $\chi_2(u)$, 4ω, $2\omega + 2\omega'$;

(iii) for $\chi_3(u)$, 4ω, $2\omega'$. These facts can be immediately verified by the ordinary formulæ of elliptic functions.

Further, on the surface thus dissected the function

$$\mathrm{sn}\,(u\sqrt{e_1-e_3}),\ =\sqrt{e_1-e_3}/\sqrt{x-e_3}$$

does not take all values, because the values of the argument u and $-u-2\omega$ give the same values for the function. To get all values we take two exemplars of the Riemann surface, defining the sign of $\sqrt{x-e_3}$ differently in these, and join them into one surface by making the loop (a) a cross line between them, so that the left edge of this loop in one exemplar becomes continuous with the right edge of this loop in the other exemplar, and *vice versa.*



In other words, if we take as the fundamental surface a parallelogram with 2ω, $2\omega'$ as contiguous sides, so that the opposite sides AD, BC represent respectively the right and left edges of the period loop (a), the function $\mathrm{sn}\,(u\sqrt{e_1-e_3})$ does not take all values within this parallelogram because the values in the triangle ACD are equal to the values in the triangle CAB; to get all values we must take another equal parallelogram wherein the value of $\mathrm{sn}\,(u\sqrt{e_1-e_3})$ at any place is the negative of the value of $\mathrm{sn}\,(u\sqrt{e_1-e_3})$ at the corresponding place of the first parallelogram, and then join BC with $A'D'$ and AD with $B'C'$. We thus get the ordinary period parallelogram with $2\omega'$ and 4ω as sides, in which $\mathrm{sn}\,(u\sqrt{e_1-e_3})$ takes every value twice over. We may then also join $DCD'C'$ with $ABA'B'$, so obtaining an anchor ring upon which $\mathrm{sn}\,(u\sqrt{e_1-e_3})$ is single-valued, but takes every value twice over.

Another point may be noticed in this elliptic case; similar remarks apply in general; taking the dissection as before, and supposing e_1, e_2, e_3 to be real, and further supposing that for real large positive values of x, in the lower sheet of the surface, the functions $\sqrt{x-e_1}$, $\sqrt{x-e_2}$, $\sqrt{x-e_3}$ are positive, as well as real, then the half-period

$$\omega=\int_{e_1}^{\infty}\frac{dx}{y}=\int_{e_1}^{\infty}\frac{dx}{2\sqrt{x-e_1}\sqrt{x-e_2}\sqrt{x-e_3}}$$

is *real and positive*, while the half-period

$$\omega' = \int_{e_3}^{e_1} \frac{dx}{y}$$

is $+\,i$ *times a real positive* quantity; thus $R\left(\frac{\omega'}{i\omega}\right)$ is positive, while, as we have seen, $\sqrt{e_3 - e_1} = -\,i\sqrt{e_1 - e_3}$, etc. If, however, we had taken a dissection such as (II) we should have obtained $R\left(\frac{\omega'}{i\omega}\right) =$ negative, and, correspondingly, $\sqrt{e_3 - e_1} = +\,i\sqrt{e_1 - e_3}$, etc.

APPENDIX II TO SECTION IV.—*On a certain prime function.*

If $\frac{1}{2}\Omega$ be an odd half-period such that (cf. B. 302)

$$\tfrac{1}{2}\Omega = \tfrac{1}{2}(s + \tau s') = v^{a_p,\,a} - v^{n_1,\,a_1} - \ldots\ldots - v^{n_{p-1},\,a_{p-1}},$$

then (B. 427) we have, in the function,

$$\varpi(x, z) = \frac{\Theta\left(v^{x,\,z} + \tfrac{1}{2}\Omega\right) e^{\pi i s' v^{x,\,z}}}{\sqrt{(x - n_1)\ldots\ldots(x - n_{p-1})(z - n_1)\ldots\ldots(z - n_{p-1})}} = C\,\frac{(x - z)\, e^{\frac{1}{2}\Pi_{x,\,z}^{\bar{x},\,\bar{z}}}}{\sqrt{ys}},$$

wherein C is a certain constant, and $\bar{x}$, $\bar{z}$ denote the places conjugate to x and z, a function which, as appears from the first form ($n_1, \ldots\ldots, n_{p-1}$ being branch places), is single-valued on the dissected Riemann surface, and has no infinities; the function vanishes to the first order when x is at z, and to order $p - 1$, as a function of x, when x is at infinity. Replacing any factor $x - n_i$ by $x_1 - n_i x_2$, where $x = x_1/x_2$, we have a prime *form*, with no infinities, and only one zero, at $x = z$. We consider, however, only the function. After what has been given, it is easy to determine the factors of this, regarded as a function of x, at the period loops. For if

$$v^{a,\,n_r} = \tfrac{1}{2}(\alpha_r + \tau\alpha_r'),$$

so that

$$\sqrt{x - n_r} \propto \begin{pmatrix}\alpha_r \\ \alpha_r'\end{pmatrix},$$

then

$$\tfrac{1}{2}\Omega = v^{a,\,n_1} + \ldots\ldots + v^{a,\,n_{p-1}} - v^{a,\,a_1} - \ldots\ldots - v^{a,\,a_p}$$

$$= \tfrac{1}{2}\begin{pmatrix}\Sigma\alpha_r' \\ \Sigma\alpha_r\end{pmatrix} - \tfrac{1}{2}\begin{pmatrix}1, & 1 & , \ldots\ldots, & 1 \\ p, & p-1 & , \ldots\ldots, & 1\end{pmatrix},$$

where the dissection is supposed to be of the kind (I); hence

$$s = \Sigma a_r - (p, p-1, \ldots, 1), \quad s' = \Sigma a_r' - (1, 1, \ldots, 1).$$

Thus the factor at the k^{th} (a) loop

$$= \frac{e^{\pi i s_k'}}{e^{\pi i \Sigma a_r'}}, = -1;$$

and the factor at the k^{th} period loop of the second kind is

$$\frac{e^{-2\pi i (v_k^{x, z} + \frac{1}{2}\tau_{k,k})} e^{-\pi i s_k}}{e^{\pi i \Sigma a_r}} = (-1)^{p-k+1} e^{-2\pi i (v_k^{x, z} + \frac{1}{2}\tau_{k,k})}.$$

In particular we may take $n_1, n_2, \ldots, n_{p-1} = a_1, a_2, \ldots, a_{p-1}$, and $\frac{1}{2}\Omega = u^{a, a_p}$; then, with the dissection (I), *the function of x given by*

$$\varpi(x, z) = \frac{\Theta\left(v^{x, z} \,\middle|\, \tfrac{1}{2}\begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 1 & \ldots & 1 & 1 \end{pmatrix}\right)}{\sqrt{(x-a_1)\ldots(x-a_{p-1})}\sqrt{(z-a_1)\ldots(z-a_{p-1})}}$$

has the following properties: (i) *it vanishes when the place x is at the place z, to the first order, and vanishes $p-1$ times at infinity;* (ii) *it has no infinity;* (iii) $\varpi(x, z) = -\varpi(z, x)$; (iv) *at every loop of the first kind it has the factor* -1; *at the k^{th} loop of the second kind it has the factor*

$$(-1)^{p-k+1} e^{-2\pi i (v_k^{x, z} + \frac{1}{2}\tau_{k,k})}.$$

In the elliptic case the function is the odd theta function; in case $p = 2$ it is

$$\Theta\left(v^{x, z} \,\middle|\, \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right) \Big/ \sqrt{(x-a_1)(z-a_1)}.$$

SECTION V.

On the construction of the function ϑ.

If $R_{z, c}^{x, k}$ be an elementary integral of the third kind, with z, c as infinities, chosen so as to vanish at the place k, and be further such that $R_{z, c}^{x, k} = R_{x, k}^{z, c}$, then there will be an equation of the form

$$R_{z, c}^{x, k} = \Pi_{z, c}^{x, k} - 2\sum_{i=1}^{p}\sum_{j=1}^{p} a_{ij} u_i^{x, k} u_j^{z, c},$$

wherein $\Pi_{z, c}^{x, k}$ is Riemann's normal elementary integral of the third kind,

$u_1^{x,k}, \ldots, u_p^{x,k}$ are any linearly independent integrals of the first kind, and the coefficients $a_{i,j}$ are constants such that $a_{i,j}=a_{j,i}$. Taking then arguments $u_1, \ldots, u_p$, of which the periods of u_r are $2\omega_{r,1}, \ldots, 2\omega_{r,p}, 2\omega'_{r,1}, \ldots, 2\omega'_{r,p}$, these being supposed to satisfy the necessary relations (B. 197, 285), and determining the arguments $v_1, \ldots, v_p$ and the matrix τ by the equations denoted by

$$u = 2\omega v, \quad \omega' = \omega\tau,$$

the function $\vartheta\left(u \,\middle|\, \tfrac{1}{2}\binom{q'}{q}\right)$ is given by

$$\vartheta\left(u \,\middle|\, \tfrac{1}{2}\binom{q'}{q}\right) = \Sigma e^{au^2 + 2\pi i v(n+\frac{1}{2}q') + i\pi\tau(n+\frac{1}{2}q')^2 + i\pi q(n+\frac{1}{2}q')},$$

wherein $au^2 = \Sigma\Sigma a_{ij} u_i u_j$, etc., and the summation is in regard to each of $n_1, \ldots, n_p$ from $-\infty$ to $+\infty$. Thus the function is equal to $e^{au^2}\Theta\left(v \,\middle|\, \tfrac{1}{2}\binom{q'}{q}\right)$. Associated with the functions $\vartheta(u)$ there are, beside the periods 2ω, $2\omega'$, certain other quantities 2η, $2\eta'$ which are the negatives of the periods of the integrals of the second kind $L_1^{x,a}, \ldots, L_p^{x,a}$ which arise (B. 194) in the equation

$$R_{z,c}^{x,k} = \int_k^x \frac{dx}{2y}\left(\frac{y+s}{x-z} - \frac{y+d}{x-c}\right) + \sum_{i=1}^{p} u_i^{x,k} L_i^{z,c},$$

wherein (z, s), (c, d) are the values of (x, y) when x is at the places z, c respectively.

The general form of $R_{z,c}^{x,k}$ in the hyperelliptic case is

$$R_{z,c}^{x,k} = \int_k^x \int_c^z \frac{dx}{y} \cdot \frac{dz}{s} \cdot \frac{2ys + F(x, z)}{4(x-z)^2},$$

where $F(x, z)$ is any rational integral polynomial, symmetrical in x, z, of order $p+1$ in each, which satisfies the two conditions (B. 315)

$$F(z, z) = 2f(z), \quad \left[\frac{\partial}{\partial x} F(x, z)\right]_{x=z} = \frac{d}{dz} f(z);$$

in case the fundamental algebraic equation be

$$y^2 = \lambda + \lambda_1 x + \ldots + \lambda_{2p+1} x^{2p+1} + \lambda_{2p+2} x^{2p+2} = f(x),$$

it can easily be verified that a possible form for the function $F(x, z)$ is that given by

$$f(x, z) = \sum_{i=0}^{p+1} x^i z^i \left[2\lambda_{2i} + \lambda_{2i+1}(x+z)\right];$$

and any other form for $F(x, z)$ differs from $f(x, z)$ by an expression of the form

$$(x-z)^2(x, z)_{p-1},$$

wherein $(x, z)_{p-1}$ denotes an integral symmetrical polynomial of order $p-1$ in each of x and z; conversely, the $\frac{1}{2}p(p+1)$ coefficients in $(x, z)_{p-1}$ are the arbitrary constants of the integral $R^{x, k}_{z, c}$; by the suitable choice of these the quantities a_{ij} in the definition of the functions ϑ may be made to take any assigned values.

Another possible form for the function $F(x, z)$, than the one $f(x, z)$ above given, is that given by

$$4[P(x)Q(z)+P(z)Q(x)],$$

where

$$P(x)=(x-a_1)\ldots\ldots(x-a_p),\quad Q(x)=(x-c_1)\ldots\ldots(x-c_p)(x-c).$$

Still another form for $F(x, z)$, which is of great importance, may be most shortly defined thus: in the equation $y^2=f(x)$ put $x=x_1/x_2$, and write it in the form $y^2x_2^{2p+2}=a_x^{2p+2}$, the notation on the right being the ordinary symbolical notation for binary forms; then a possible form for $F(x, z)$ is that given by $2a_x^{p+1}a_z^{p+1}/x_2^{p+1}z_2^{p+1}$. The advantage of this form is that the resulting integral $R^{x, k}_{z, c}$ is covariantive under cogredient linear substitutions of the variables x, z (cf. for example Klein, Math. Annal., XXVII (1886), p. 457).

In what follows we shall, unless the contrary is stated, suppose the arbitrary coefficients in the polynomial $F(x, z)$ to be left undetermined. Writing then the expansion

$$\frac{\vartheta(u)}{\vartheta(0)}=1+\sum_{i=1}^{p}\sum_{j=1}^{p}c_{i,j}u_iu_j+\ldots\ldots,$$

we may suppose the coefficients $c_{i,j}$ to be at our disposal.

SECTION VI.

The expression of the quotients of the ϑ functions with one suffix by means of algebraic functions.

In this section we suppose the Riemann surface dissected in one of the ways (I), (II), (III), (IV). We shall however ultimately adopt (I) as the normal

method of dissection. We use the following notations, other than those already introduced:

$$P(x) = (x - a_1) \ldots\ldots (x - a_p), \quad Q(x) = (x - c_1) \ldots\ldots (x - c_p)(x - c);$$
$$y^2 = 4P(x)\,Q(x), \; = f(x);$$
$$F(x) = (x - x_1)(x - x_2) \ldots\ldots (x - x_p);$$
$$u_i = u_i^{x_1, a_1} + u_i^{x_2, a_2} + \ldots\ldots + u_i^{x_p, a_p}, \qquad (i = 1, 2, \ldots\ldots, p)$$

wherein
$$u_i^{x, k} = \int_k^x \frac{x^{i-1}dx}{y};$$
$$V_i = V_i^{x_1, a_1} + V_i^{x_2, a_2} + \ldots\ldots + V_i^{x_p, a_p}, \qquad (i = 1, 2, \ldots\ldots, p)$$

where
$$V_i^{x, k} = \frac{\sqrt{f'(a_i)}}{P'(a_i)} \int_k^x \frac{P(x)}{x - a_i} \frac{dx}{2y}$$

and
$$\sqrt{f'(a_i)} = 2\sqrt{a_i - c_1}\sqrt{a_i - a_1}\sqrt{a_i - c_2} \ldots\ldots \sqrt{a_i - c},$$
$$P'(a_i) = (a_i - a_1)(a_i - a_2) \ldots\ldots (a_i - a_p);$$

then
$$u_i^{x, k} = \frac{2a_1^{i-1}}{\sqrt{f'(a_1)}} V_1^{x, k} + \ldots\ldots + \frac{2a_p^{i-1}}{\sqrt{f'(a_p)}} V_p^{x, k},$$

and
$$\frac{\partial}{\partial V_i} = \frac{2}{\sqrt{f'(a_i)}} \left(\frac{\partial}{\partial u_1} + a_i \frac{\partial}{\partial u_2} + \ldots\ldots + a_i^{p-1} \frac{\partial}{\partial u_p} \right),$$
$$(i = 1, 2, \ldots\ldots, p).$$

Further, putting

$$P(x) = x^p + h_1 x^{p-1} + \ldots\ldots + h_p,$$

we have

$$\frac{P(x)}{x - a_r} = x^{p-1} + (a_r + h_1)\,x^{p-2} + \ldots\ldots + (a_r^{p-1} + h_1 a_r^{p-2} + \ldots\ldots + h_{p-1})$$
$$= x^{p-1} + \chi_1(a_r)\,x^{p-2} + \ldots\ldots + \chi_{p-1}(a_r), \text{ say,}$$

so that
$$a_i^{p-1} + \chi_1(a_r)\,a_i^{p-2} + \ldots\ldots + \chi_{p-1}(a_r) = 0 \text{ or } P'(a_r),$$

according as $i \neq r$, or $i = r$;

thus
$$u_p + \chi_1(a_r)\,u_{p-1} + \ldots\ldots + \chi_{p-1}(a_r)\,u_1 = 2V_r \frac{P'(a_r)}{\sqrt{f'(a_r)}};$$

we shall, immediately, introduce a constant λ_r, given, for a proper determination of the sign of the denominator, by

$$\lambda_r = \frac{(-1)^{p-r}\sqrt{P'(a_r)}}{(i\sqrt{f'(a_r)/4})^{\frac{1}{2}}}\,;$$

with this we have

$$u_p + \chi_1(a_r)\,.\,u_{p-1} + \ldots\ldots + \chi_{p-1}(a_r)\,.\,u_1 = i\lambda_r^2 V_r.$$

If λ, μ, ... denote either linear functions $x-b$, $b-x$, or differences such as $b_i - b_j$, between values of x at the finite branch places, we mean by $\sqrt{\lambda\mu\ldots\ldots}$ the product $\sqrt{\lambda}\sqrt{\mu}\ldots\ldots$, where $\sqrt{\lambda}$, $\sqrt{\mu}$, ... have the values previously assigned to them. We have already introduced $\sqrt{f'(a_r)}$ defined according to this rule.

If, in the variable V_i, we suppose $x_1, \ldots\ldots, x_p$ to be respectively near to the branch places $a_1, \ldots\ldots, a_p$, and put $x_i = a_i + t_i^2$, we easily find that $V_i^{x_r,\,a_r}$ vanishes to the third order when $r \neq i$, while $V_i^{x_i,\,a_i} = t_i$; hence when $x_1, \ldots\ldots, x_p$ come to $a_1, \ldots\ldots, a_p$, the variable $V_i = t_i$.

For the integrals V_i see B. 169; they are more convenient for our purpose than Weierstrass's integrals (B. 325–6).

We find the following equations to hold for the dissection (I):

$$\begin{aligned}
\sqrt{(-1)^{2p-2r+1}f'(a_r)} &= i^{2p-2r+1}\sqrt{f'(a_r)}, & \sqrt{(-1)^{2p-2r+2}f'(c_r)} &= i^{2p-2r+2}\sqrt{f'(c_r)},\\
\sqrt{(-1)^{p-r}P'(a_r)} &= i^{p-r}\sqrt{P'(a_r)}\,, & \sqrt{(-1)^{p-r+1}P(c_r)} &= i^{p-r+1}\sqrt{P(c_r)}\,,\\
\sqrt{(-1)^{p-r+1}Q'(c_r)} &= i^{p-r+1}\sqrt{Q'(c_r)}\,, & \sqrt{(-1)^{p-r+1}Q(a_r)} &= i^{p-r+1}\sqrt{Q(a_r)}\,,\\
\sqrt{(-1)^{p-r+1}F(a_r)} &= (-i)^{p-r+1}\sqrt{F(a_r)}, & \sqrt{(-1)^{p-r+1}F(c_r)} &= (-i)^{p-r+1}\sqrt{F(c_r)},
\end{aligned}$$

where in each case, except the last two, the expression under the square root on the left side is merely an abbreviation for a product in which each factor is a *positive* difference, that is, for a difference in which the branch value subtracted has a lower position in the ascending order of the branch places than the branch value from which it is subtracted; for instance, in these equations, $(-1)^{p-r}P'(a_r)$ is an abbreviation for $(a_r - a_1)\ldots\ldots(a_r - a_{r-1})(a_{r+1} - a_r)\ldots\ldots(a_p - a_r)$. The same equations hold for the dissection (III). In the cases of the dissections (II), (IV), the i occurring on the right side of these equations is to be replaced by $-i$. The expressions $(-1)^{p-r+1}F(a_r)$, $(-1)^{p-r+1}F(c_r)$, in the last two of the equations above, are abbreviations for

$$(a_r - x_1)\ldots\ldots(a_r - x_{r-1})(x_r - a_r)\ldots\ldots(x_p - a_r),$$
$$(c_r - x_1)\ldots\ldots(c_r - x_{r-1})(x_r - c_r)\ldots\ldots(x_p - c_r),$$

respectively; they are introduced merely to enable us to write the formulæ more conveniently. In the standard case, when the branch values $c_1, a_1, \ldots$ are real quantities in ascending order of magnitude, the expressions occurring on the left sides in these equations, except the last two, are real quantities.

Consider now the functions

$$cl_r(u) = \frac{\vartheta(u \mid u^{a, c_r})}{\vartheta(u)} \Big/ \frac{\vartheta(0 \mid u^{a, c_r})}{\vartheta(0)}, \quad al_r(u) = \frac{\vartheta(u \mid u^{a, a_r})}{\vartheta(u)} \Big/ \frac{\vartheta'_r(0 \mid u^{a, a_r})}{\vartheta(0)},$$

where

$$\vartheta'_r(u \mid u^{a, a_r}) = \frac{\partial}{\partial V_r} \vartheta(u \mid u^{a, a_r}).$$

The quotient

$$cl_r(u) \Big/ \frac{\sqrt{x_1 - c_r} \ldots \sqrt{x_p - c_r}}{\sqrt{a_1 - c_r} \ldots \sqrt{a_p - c_r}}$$

is, on the dissected Riemann surface, a single-valued continuous function of each of $x_1, \ldots, x_p$, without zeros or infinities (B. 309); its value when the arguments u vanish, that is, when $x_1 = a_1, \ldots, x_p = a_p$, is $+1$. As the square of this function is $+1$, the function itself is $+1$. Thus, *in the case of the dissections* (I), (III), we have

$$cl_r(u) = \frac{(-i)^p \sqrt{F(c_r)}}{(-i)^{r-1}(i)^{p-r+1}\sqrt{P(c_r)}} = (-1)^{p-r+1}\frac{\sqrt{F(c_r)}}{\sqrt{P(c_r)}}$$
$$= \frac{\sqrt{(-1)^{p-r+1}F(c_r)}}{\sqrt{(-1)^{p-r+1}P(c_r)}}$$

(cf. Weierstrass, Math. Werke, I, 1894, p. 330, remembering that the branch places are here numbered in the reverse order. Cf. B. 569).

The function

$$al_r(u) = \frac{\vartheta(u \mid u^{a, a_r})}{\vartheta(u)} \Big/ \frac{\vartheta'_r(0 \mid u^{a, a_r})}{\vartheta(0)},$$

when $x_1, \ldots, x_p$ approach respectively to $a_1, \ldots, a_p$, is equal to t_r; hence the quotient

$$al_r(u) \Big/ \frac{\sqrt{x_1 - a_r} \ldots \sqrt{x_p - a_r}}{\sqrt{a_1 - a_r} \ldots \sqrt{a_p - a_r}},$$

wherein, in the denominator, there are $p - 1$ factors $\sqrt{a_i - a_r}$, is (B. 309) equal to $+1$, namely, for a dissection (I) or (III),

$$al_r(u) = \frac{(-i)^p \sqrt{F(a_r)}}{(-i)^{r-1}(i)^{p-r}\sqrt{P'(a_r)}} = (-1)^{p-r+1} i \frac{\sqrt{F(a_r)}}{\sqrt{P'(a_r)}},$$
$$= \frac{\sqrt{(-1)^{p-r+1}F(a_r)}}{\sqrt{(-1)^{p-r}P'(a_r)}}.$$

From these results we can deduce values for the quotient

$$\frac{\vartheta(u \mid u^{a, c_r})}{\vartheta(u)}, \quad \frac{\vartheta(u \mid u^{a, a_r})}{\vartheta(u)};$$

we limit ourselves to the dissection (I).

We have $\qquad u_r^{c_1, a_1} + u_r^{c_2, a_2} + \ldots + u_r^{c, a} = 0, \qquad (r = 1, 2, \ldots, p)$

and (B. 285)

$$\frac{\vartheta\left(u + \tfrac{1}{2}\Omega_q \,\middle|\, \tfrac{1}{2}\binom{q'}{q}\right)}{\vartheta(u + \tfrac{1}{2}\Omega_q)} = e^{-\frac{1}{2}\pi i q' q} \frac{\vartheta(u)}{\vartheta\left(u \,\middle|\, \tfrac{1}{2}\binom{q'}{q}\right)};$$

if, therefore, in the expression obtained for $cl_r(u)$, we put $x_1, \ldots, x_p$ respectively at $c_1, \ldots, c_{r-1}, c, c_{r+1}, \ldots, c_p$, in which case $u_i = u_i^{a, c_r}$, we obtain

$$\frac{\vartheta^2(0)}{\vartheta^2(0 \mid u^{a, c_r})} = (-1)^{p-r+1} i^p \frac{\sqrt{(c_1 - c_r)(c_2 - c_r)\ldots(c_{r-1} - c_r)(c - c_r)(c_{r+1} - c_r)\ldots(c_p - c_r)}}{\sqrt{P(c_r)}}$$

$$= (-1)^{p-r+1} i^p (-i)^{r-1} (i)^{p-r+1} \frac{\sqrt{Q'(c_r)}}{\sqrt{P(c_r)}}$$

$$= \frac{\sqrt{Q'(c_r)}}{\sqrt{P(c_r)}} = \frac{\sqrt{(-1)^{p-r+1} Q'(c_r)}}{\sqrt{(-1)^{p-r+1} P(c_r)}}$$

$$= \frac{\sqrt{f'(c_r)/4}}{P(c_r)} = (-1)^{p-r+1} \frac{\sqrt{(-1)^{2p-2r+2} f'(c_r)/4}}{P(c_r)};$$

from this equation we define one of the square roots of $\sqrt{f'(c_r)/4}$ by the equation

$$(\sqrt{f'(c_r)/4})^{\frac{1}{2}} = (-1)^{p-r+1} \sqrt{P(c_r)}\, \vartheta(0)/\vartheta(0 \mid u^{a, c_r}),$$

and hence obtain

$$\frac{\vartheta(u \mid u^{a, c_r})}{\vartheta(u)} = \frac{\sqrt{F(c_r)}}{(\sqrt{f'(c_r)/4})^{\frac{1}{2}}},$$

and we put similarly

$$(\sqrt{(-1)^{2p-2r+2} f'(c_r)/4})^{\frac{1}{2}} \quad \text{or} \quad (i^{2p-2r+2}\sqrt{f'(c_r)})^{\frac{1}{2}}$$

equal to $\qquad (-i)^{p-r+1}(\sqrt{f'(c_r)/4})^{\frac{1}{2}};$

that is $\qquad \sqrt{(-1)^{p-r+1} P(c_r)}\, \vartheta(0)/\vartheta(0 \mid u^{a, c_r})$

and so obtain
$$\frac{\vartheta(u\,|\,u^{a,\,c_r})}{\vartheta(u)}=\frac{\sqrt{(-1)^{p-r+1}F(c_r)}}{\left(\sqrt{(-1)^{2p-2r+2}f'(c_r)/4}\right)^{\frac{1}{2}}}.$$

Cf. Weierstrass, Math. Werke, I, 1894, p. 135.

Next, if in $al_r(u)$ we put $x_1,\ldots,x_p$ respectively equal to $a_1,\ldots,a_{r-1},a,a_{r+1},\ldots,a_p$, and therefore u equal to $u^{a,\,a_r}$, we obtain
$$al_r(u)=\frac{1}{i}\left[\frac{\vartheta(0)}{\vartheta_r'(0\,|\,u^{a,\,a_r})}\right]^2\frac{1}{(DV_r)_a t},$$

where $(DV_r)_a t$ is the value when $x=t^{-2}$, and t is small, of the integral
$$\frac{\sqrt{f'(a_r)}}{P'(a_r)}\int_a^x\frac{P(x)}{x-a_r}\,\frac{dx}{2y},$$

and is therefore equal to
$$-\frac{\sqrt{f'(a_r)/4}}{P'(a_r)}\,t;$$

hence we have
$$\begin{aligned}&i\left[\frac{\vartheta(0)}{\vartheta_r'(0)\,|\,u^{a,\,a_r})}\right]^2\frac{P'(a_r)}{\sqrt{f'(a_r)/4}}\cdot\frac{1}{t}\\&\quad=\frac{1}{t}\,i(-1)^{p-r+1}i^p\,\frac{\sqrt{(a_1-a_r)\ldots(a_{r-1}-a_r)(a_{r+1}-a_r)\ldots(a_p-a_r)}}{\sqrt{P'(a_r)}}\\&\quad=\frac{1}{t}\,i(-1)^{p-r+1}i^p(-i)^{r-1}(i)^{p-r}=\frac{1}{t}.\end{aligned}$$

We define now one square root of $i\sqrt{f'(a_r)/4}$ by the equation
$$\left(i\sqrt{f'(a_r)/4}\right)^{\frac{1}{2}}=i(-1)^{p-r}\sqrt{P'(a_r)}\,\vartheta(0)/\vartheta_r'(0\,|\,u^{a,\,a_r}),$$

and hence obtain
$$\frac{\vartheta(u\,|\,u^{a,\,a_r})}{\vartheta(u)}=\frac{\sqrt{F(a_r)}}{\left(i\sqrt{f'(a_r)/4}\right)^{\frac{1}{2}}};$$

further, we put
$$\left(\sqrt{(-1)^{2p-2r+1}f'(a_r)/4}\right)^{\frac{1}{2}}_r\quad\text{or}\quad\left(i^{2p-2r+1}\sqrt{f'(a_r)/4}\right)^{\frac{1}{2}}$$

equal to $\quad(-i)^{p-r}\left(i\sqrt{f'(a_r)/4}\right)^{\frac{1}{2}},$

that is, to $\quad i^{p-r+1}\sqrt{P'(a_r)}\,\vartheta(0)/\vartheta_r'(0\,|\,u^{a,\,a_r}),$

or $\quad i\sqrt{(-1)^{p-r}P'(a_r)}\,\vartheta(0)/\vartheta_r'(0\,|\,u^{a,\,a_r}),$

and so obtain
$$\frac{\vartheta(u\,|\,u^{a,\,a_r})}{\vartheta(u)}=i\,\frac{\sqrt{(-1)^{p-r+1}F(a_r)}}{\left(\sqrt{(-1)^{3p-2r+1}f'(a_r)/4}\right)^{\frac{1}{2}}}.$$

Cf. Weierstrass, Math. Werke, I, 1894, p. 135.

The results thus obtained may be summarized as follows:

Let
$$\lambda_r=\frac{(-1)^{p-r}\sqrt{P'(a_r)}}{\left(i\sqrt{f'(a_r)/4}\right)^{\frac{1}{2}}}=\frac{\sqrt{(-1)^{p-r}P'(a_r)}}{\left(\sqrt{(-1)^{3p-3r+1}f'(a_r)/4}\right)^{\frac{1}{2}}},$$

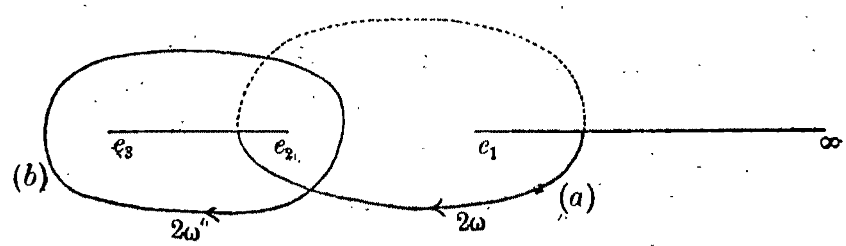
where the signification of the denominator is determined by
$$\frac{\vartheta_r'(0\,|\,u^{a,\,a_1})}{\vartheta(0)}=i\lambda_r;$$

also let
$$\mu_r=\frac{(-1)^{p-r+1}\sqrt{P(c_r)}}{\left(\sqrt{f'(c_r)/4}\right)^{\frac{1}{2}}}=\frac{\sqrt{(-1)^{p-r+1}P(c_r)}}{\left(\sqrt{(-1)^{2p-2r+3}f'(c_r)/4}\right)^{\frac{1}{2}}},$$

so that
$$\frac{\vartheta(0\,|\,u^{a,\,c_r})}{\vartheta(0)}=\mu_r;$$

then we have
$$\frac{\vartheta(u\,|\,u^{a,\,a_r})}{\vartheta(u)}=(-1)^{p-r}\lambda_r\frac{\sqrt{F(a_r)}}{\sqrt{P'(a_r)}},\quad \frac{\vartheta(u\,|\,u^{a,\,c_r})}{\vartheta(u)}=(-1)^{p-r+1}\mu_r\frac{\sqrt{F(c_r)}}{\sqrt{P(c_r)}}.$$

APPENDIX TO SECTION VI.—*The Elliptic case.*



Replacing c_1, a_1, c respectively by e_3, e_2, e_1 we have
$$u=u^{x,\,e_3}=u^{\infty,\,e_3}-u^{\infty,\,x}=\omega+\omega'-U,\ \text{say},$$

$$\vartheta(u\,|\,u^{a,\,c_1})=\vartheta\left(\omega+\omega'-U\,\middle|\,\tfrac{1}{2}\binom{1}{0}\right)=\quad\varepsilon\vartheta\left(-U\,\middle|\,\tfrac{1}{2}\binom{0}{1}\right)=\quad\varepsilon\vartheta_0\left(\frac{U}{2\omega}\right),$$
$$\vartheta(u\,|\,u^{a,\,a_1})=\vartheta\left(\omega+\omega'-U\,\middle|\,\tfrac{1}{2}\binom{1}{1}\right)=\quad\frac{1}{i}\,\varepsilon\vartheta\left(-U\,\middle|\,\tfrac{1}{2}\binom{0}{0}\right)=\frac{1}{i}\,\varepsilon\vartheta_3\left(\frac{U}{2\omega}\right),$$
$$\vartheta(u\,|\,u^{a,\,c})=\vartheta\left(\omega+\omega'-U\,\middle|\,\tfrac{1}{2}\binom{0}{1}\right)=-\frac{1}{i}\,\varepsilon\vartheta\left(-U\,\middle|\,\tfrac{1}{2}\binom{1}{0}\right)=\quad i\varepsilon\vartheta_2\left(\frac{U}{2\omega}\right),$$
$$\vartheta(u)\quad=\vartheta\left(\omega+\omega'-U\right)\quad=\quad\varepsilon\vartheta\left(-U\,\middle|\,\tfrac{1}{2}\binom{1}{1}\right)=\quad\varepsilon\vartheta_1\left(\frac{U}{2\omega}\right),$$

where the notation is that of Weierstrass (cf. Formeln u. Lehrsätze, or Halphen, Fonct. Ellipt., I, p. 252), and ε is $e^{-(\eta+\eta')(U-\frac{1}{2}\omega-\frac{1}{2}\omega')}$; also, if $u/2\omega=\nu$,

$$\left[\frac{\partial}{\partial V}\vartheta\left(u\,|\,u^{a,\,a_1}\right)\right]_{u=0}=\frac{1}{2\omega\sqrt{(e_2-e_3)(e_2-e_1)}}\left[-\frac{\partial}{\partial\nu}\vartheta\left(u\,\middle|\,\tfrac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\right)\right]_{u=0}$$

$$=-\frac{\vartheta_1'}{2\omega\sqrt{(e_2-e_3)(e_2-e_1)}},$$

where ϑ_1' is the quantity so denoted by Halphen, p. 259.

Therefore our results give.

$$\frac{-i\vartheta_1'/2\omega\sqrt{(e_2-e_3)(e_1-e_2)}}{\vartheta_3}=i\lambda_1=\frac{i}{\left(i\sqrt{(e_2-e_3)(e_2-e_1)}\right)^{\frac{1}{2}}}=\frac{i}{\left(\sqrt{(e_2-e_3)(e_1-e_2)}\right)^{\frac{1}{2}}},$$

$$\frac{\vartheta_2}{\vartheta_3}=\mu_1=-\frac{\sqrt{e_3-e_2}}{\left(\sqrt{(e_3-e_2)(e_3-e_1)}\right)^{\frac{1}{2}}}=\frac{\sqrt{e_2-e_3}}{\left(\sqrt{(e_2-e_3)(e_1-e_3)}\right)^{\frac{1}{2}}},$$

$$\frac{\vartheta_0}{\vartheta_3}=\mu_2=\frac{\sqrt{e_1-e_2}}{\left(\sqrt{(e_1-e_2)(e_1-e_3)}\right)^{\frac{1}{2}}}$$

and

$$\frac{1}{i}\,\frac{\vartheta_3(\nu)}{\vartheta_1(\nu)}=\lambda_1\sqrt{e_2-x}\quad=\frac{i}{\left(\sqrt{(e_2-e_3)(e_1-e_2)}\right)^{\frac{1}{2}}}\sqrt{x-e_2},$$

$$\frac{\vartheta_0(\nu)}{\vartheta_1(\nu)}=-\mu_1\frac{\sqrt{e_3-x}}{\sqrt{e_3-e_2}}=\mu_1\frac{\sqrt{x-e_3}}{\sqrt{e_2-e_3}}=\frac{1}{\left(\sqrt{(e_2-e_3)(e_1-e_3)}\right)^{\frac{1}{2}}}\sqrt{x-e_3},$$

$$i\,\frac{\vartheta_2(\nu)}{\vartheta_1(\nu)}=\mu_2\frac{\sqrt{e_1-x}}{\sqrt{e_1-e_2}}\quad=\frac{i}{\left(\sqrt{(e_1-e_2)(e_1-e_3)}\right)^{\frac{1}{2}}}\sqrt{x-e_1}.$$

Comparing the forms for ϑ_1'/ϑ_3, ϑ_2/ϑ_3, ϑ_0/ϑ_3 with those given by Halphen, Fonct. Elliptiq., I, p. 258, we immediately see that the determinations adopted by us for the square roots are given by

$$\left(\sqrt{(e_2-e_3)(e_1-e_2)}\right)^{\frac{1}{2}}=-\sqrt[4]{e_2-e_3}\sqrt[4]{e_1-e_2},$$

$$\left(\sqrt{(e_2-e_3)(e_1-e_3)}\right)^{\frac{1}{2}}=\sqrt[4]{e_2-e_3}\sqrt[4]{e_1-e_3},$$

$$\left(\sqrt{(e_1-e_2)(e_1-e_3)}\right)^{\frac{1}{2}}=\sqrt[4]{e_1-e_2}\sqrt[4]{e_1-e_3},$$

and that, with these determinations, the values found from our formulæ for the quotients $\vartheta_3(\nu)/\vartheta_1(\nu)$, $\vartheta_0(\nu)/\vartheta_1(\nu)$, $\vartheta_2(\nu)/\vartheta_1(\nu)$ agree with those given by Halphen, p. 260.

It is to be remarked that the formulæ of Halphen contain one, namely, $\sqrt[4]{e_1-e_3}=\sqrt{\frac{\pi}{2\omega}}\,\vartheta_3$, of which we have not put down the generalization. This is given by Thomae, Crelle, LXXI (1870), p. 216.

SECTION VII.

The expression of ϑ functions of two or more suffixes algebraically.

If in general the quotient $\vartheta(u|u^{a,b_1}+\ldots+u^{a,b_k})/\vartheta(u)$, where $b_1,\ldots,b_k$ are any finite branch places, and the characteristic is *not* reduced, be denoted by $q_{b_1b_2\ldots b_k}(u)$, and for $k>1$, the quotient, of two determinants of the p^{th} order,

$$\frac{\left|\dfrac{x_r^\lambda y_r}{\phi(x_r)},\ldots,\dfrac{y_r}{\phi(x_r)},\,x_r^\mu,\ldots,1\right|}{\left|x_r^{p-1},\ldots,x_r^\mu,\ldots,1\right|},$$

wherein $\lambda=\frac{1}{2}k-1$ or $\frac{1}{2}(k-3)$, $\mu=p-1-\frac{1}{2}k$ or $p-\frac{1}{2}(k+1)$, according as k is even or odd, and $\phi(x)=(x-b_1)\ldots(x-b_k)$ be denoted by $\pi_{b_1\ldots b_k}$; then (B. 312) we have

$$q_{b_1b_2}(u)=C_{q_{b_1}q_{b_2}}\pi_{b_1b_2},$$

namely,

$$\frac{\vartheta(u|u^{a,b_1}+u^{a,b_2})\,\vartheta(u)}{\vartheta(u|u^{a,b_1})\,\vartheta(u|u^{a,b_2})}=C\,\frac{\left|\dfrac{y_r}{\phi(x_r)},\,x_r^{p-2},\ldots,x_r,\,1\right|}{\left|x_r^{p-1},\,x_r^{p-2},\ldots,x_r,\,1\right|},$$
$$=C\sum_{r=1}^{p}\frac{y_r}{(x_r-b_1)(x_r-b_2)\,F'(x_r)},$$

where C is a constant. We proceed to determine the value of C^2.

Let $u^{a,b_2}=\frac{1}{2}\binom{q'}{q}$, $u^{a,b_1}=\frac{1}{2}\binom{k'}{k}$, the notation being as previously explained. Then if $v=u+u^{a,b_1}+\Omega_{m,m'}$, where $\Omega_{m,m'}$ denotes a period, we have (B. 286)

$$\frac{\vartheta(v|u^{a,b_2})}{\vartheta(v)}=e^{\pi i(mq'-m'q)}\frac{\vartheta(u+u^{a,b_1}|u^{a,b_2})}{\vartheta(u+u^{a,b_1})}$$
$$=e^{\pi i(mq'-m'q)}e^{-\frac{1}{2}\pi ik'q}\frac{\vartheta(u|u^{a,b_1}+u^{a,b_2})}{\vartheta(u|u^{a,b_1})},$$

where the characteristic denoted by $u^{a,b_1}+u^{a,b_2}$ is unreduced, namely, is $\frac{1}{2}\binom{q'+k'}{q+k}$; therefore

$$\frac{\vartheta(u\,|\,u^{a,b_1}+u^{a,b_2})\,\vartheta(u)}{\vartheta(u\,|\,u^{a,b_1})\,\vartheta(u\,|\,u^{a,b_2})}=e^{\pi i(mq'-m'q)}e^{\frac{1}{2}\pi ik'q}\,\frac{\vartheta(v\,|\,u^{a,b_2})}{\vartheta(v)}\Big/\frac{\vartheta(u\,|\,u^{a,b_2})}{\vartheta(u)};$$

and if

$$\begin{aligned} v&=u^{z_1,a_1}+\dots+u^{z_p,a_p},\\ u&=u^{x_1,a_1}+\dots+u^{x_p,a_p},\\ F(x)&=(x-x_1)\dots(x-x_p),\\ G(x)&=(x-z_1)\dots(x-z_p),\end{aligned}$$

the right side is equal (Section VI) to

$$e^{\pi i(mq'-m'q)}\,e^{\frac{1}{2}\pi ik'q}\,\frac{\sqrt{G(b_2)}}{\sqrt{F(b_2)}}.$$

Now from $\qquad u^{z_1,x_1}+\dots+u^{z_p,x_p}+u^{b_1,a}=\Omega_{m,m'},$

it follows that $z_1,\dots,z_p,b_1$ are the zeros, and $x_1,\dots,x_p,a$ are the poles of a rational function, therefore given by

$$\frac{y}{F(x)}+(x-b_1)\sum_{i=1}^{p}\frac{y_i}{x_i-b_1}\,\frac{1}{(x-x_i)F'(x_i)}$$

(cf. B. 318), and hence it follows that

$$\frac{G(b_2)}{F(b_2)}=(b_1-b_2)\Big[\tfrac{1}{2}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(x_i-b_2)F'(x_i)}\Big]^2,$$

and therefore that

$$\frac{\vartheta^2(u\,|\,u^{a,b_1}+u^{a,b_2})\,\vartheta^2(u)}{\vartheta^2(u\,|\,u^{a,b_1})\,\vartheta^2(u\,|\,u^{a,b_2})}=\binom{B_1}{B_2}(b_1-b_2)\Big[\tfrac{1}{2}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(x_i-b_2)F'(x_i)}\Big]^2,$$

where for $B_1=\frac{1}{2}\binom{k'}{k}$, $B_2=\frac{1}{2}\binom{q'}{q}$ we have $\binom{B_1}{B_2}=e^{\pi ik'q}$.

This equation is true in the case of either of the dissections (I), (II), (III), (IV); supposing the dissection to be of the kind (I), and that, in the ascending order of the branch places, b_2 is higher than b_1, we have $\binom{B_1}{B_2}(b_1-b_2)=b_2-b_1$. Hence, retaining the notation $u^{a,b_1}+u^{a,b_2}$ for the sum, *without reduction*, of the characteristics denoted by u^{a,b_1} and u^{a,b_2}, we may write

$$\frac{\vartheta(u\,|\,u^{a,b_1}+u^{a,b_2})\,\vartheta(u)}{\vartheta(u\,|\,u^{a,b_1})\,\vartheta(u\,|\,u^{a,b_2})}=\varepsilon_{b_1b_2}\sqrt{b_2-b_1}\,\tfrac{1}{2}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(x_i-b_2)F'(x_i)},$$

where ϵ_{b_1, b_2} is ± 1, but is perfectly definite. We do not here determine its value.

Now let b_1, b_2, b_3 be any three finite branch places, which, for definiteness, we suppose to be in ascending order. Then we can determine in a similar way the constant factor in the algebraic expression of the square of the function

$$\frac{\vartheta(u|u^{a,b_1}+u^{a,b_2}+u^{a,b_3})\,\vartheta^2(u)}{\vartheta(u|u^{a,b_1})\,\vartheta(u|u^{a,b_2})\,\vartheta(u|u^{a,b_3})}.$$

For denoting $\vartheta(u|u^{a,b_1}+u^{a,b_2}+u^{a,b_3})$ momentarily by $\vartheta_{123}(u)$, etc., and putting

$$v = u^{z_1, a_1} + \ldots\ldots + u^{z_p, a_p},$$
$$u \equiv v + u^{a, b_1},$$

where the sign $\equiv$ indicates that the expressions on the two sides may differ by a period, the function

$$\frac{\vartheta^2_{123}(u)\,\vartheta^2_1(u)}{\vartheta^2_{12}(u)\,\vartheta^2_{13}(u)} = \frac{\vartheta^2_{23}(v)\,\vartheta^2(v)}{\vartheta^2_2(v)\,\vartheta^2_3(v)}$$
$$= \binom{B_2}{B_3}(b_2-b_3)\left[\tfrac{1}{2}\sum_{i=1}^{p}\frac{s_i}{(z_i-b_2)(z_i-b_3)\,G'(z_i)}\right]^2,$$

where s_i is the value of y when x is at z_i, and $z_1, \ldots, z_p$ are zeros of the rational function

$$\frac{y}{F(x)} + (x-b_1)\sum_{i=1}^{p}\frac{y_i/(x_i-b_i)}{(x-x_i)\,F'(x_i)};$$

expressing that $z_1, \ldots, z_p$ are zeros of this function we derive that

$$\sum_{r=1}^{p}\frac{s_r}{(z_r-b_2)(z_r-b_3)\,G'(z_r)}$$
$$= -\sum_{r=1}^{p}\frac{(z_r-b_1)\,F(z_r)}{(z_r-b_2)(z_r-b_3)\,G'(z_r)}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(z_r-x_i)\,F'(x_i)}$$
$$= -\frac{1}{b_2-b_3}\sum_{i=1}^{p}\sum_{r=1}^{p}\frac{(z_r-b_1)\,F(z_r)\,y_i}{(x_i-b_1)(z_r-x_i)\,F'(x_i)\,G'(z_r)}\left(\frac{1}{z_r-b_2}-\frac{1}{z_r-b_3}\right)$$
$$= -\frac{1}{b_2-b_3}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(x_i-b_2)\,F'(x_i)}\sum_{r=1}^{p}\frac{(z_r-b_1)(x_i-b_2)\,F(z_r)}{(z_r-x_i)(z_r-b_2)\,G'(z_r)}$$
$$+\frac{1}{b_2-b_3}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(x_i-b_3)\,F'(x_i)}\sum_{r=1}^{p}\frac{(z_r-b_1)(x_i-b_3)\,F(z_r)}{(z_r-x_i)(z_r-b_3)\,G'(z_r)}$$
$$= -\frac{1}{b_2-b_3}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(x_i-b_2)\,F'(x_i)}\left[x_i-b_2+(b_2-b_1)\frac{F(b_2)}{G(b_2)}\right]$$
$$+\frac{1}{b_2-b_3}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(x_i-b_3)\,F'(x_i)}\left[x_i-b_3+(b_3-b_1)\frac{F(b_3)}{G(b_3)}\right]$$
$$= \frac{b_2-b_1}{b_3-b_2}\,\frac{F(b_2)}{G(b_2)}\,\pi_{b_1 b_2} - \frac{b_3-b_1}{b_3-b_2}\,\frac{F(b_3)}{G(b_3)}\,\pi_{b_1 b_3};$$

now we have seen that

$$4\frac{G(b_2)}{F(b_2)}=(b_1-b_2)\pi^2_{b_1 b_2}, \quad 4\frac{G(b_3)}{F(b_3)}=(b_1-b_3)\pi^2_{b_1 b_3};$$

therefore

$$\frac{1}{2}\sum_{r=1}^{p}\frac{s_r}{(z_r-b_2)(z_r-b_3)\,G'(z_r)}=-\frac{2}{b_3-b_2}\left(\frac{1}{\pi_{b_1 b_2}}-\frac{1}{\pi_{b_1 b_3}}\right)$$

$$=+\frac{2}{(b_3-b_2)\,\pi_{b_1 b_2}\pi_{b_1 b_3}}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)\,F'(x_i)}\left(\frac{1}{x_i-b_2}-\frac{1}{x_i-b_3}\right)$$

$$=-\frac{2}{\pi_{b_1 b_2}\pi_{b_1 b_3}}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(x_i-b_2)(x_i-b_3)\,F'(x_i)}$$

$$=-\frac{2\pi_{b_1 b_2 b_3}}{\pi_{b_1 b_2}\pi_{b_1 b_3}}.$$

Hence we have, for either one of the dissections (I), (II), (III), (IV),

$$\frac{\vartheta^2(u\,|\,u^{a,b_1}+u^{a,b_2}+u^{a,b_3})\,\vartheta^4(u)}{\vartheta^2(u\,|\,u^{a,b_1})\,\vartheta^2(u\,|\,u^{a,b_2})\,\vartheta^2(u\,|\,u^{a,b_3})}=\frac{\vartheta^2_{123}(u)\,\vartheta^4(u)}{\vartheta^2_1(u)\,\vartheta^2_2(u)\,\vartheta^2_3(u)}$$

$$=\frac{\vartheta^2_{123}(u)\,\vartheta^2_1(u)}{\vartheta^2_{12}(u)\,\vartheta^2_{13}(u)}\cdot\frac{\vartheta^2_{12}(u)\,\vartheta^2(u)}{\vartheta^2_1(u)\,\vartheta^2_2(u)}\cdot\frac{\vartheta^2_{13}(u)\,\vartheta^2(u)}{\vartheta^2_1(u)\,\vartheta^2_3(u)}$$

$$=\binom{B_2}{B_3}(b_2-b_3)\binom{B_1}{B_2}(b_1-b_2)\binom{B_1}{B_3}(b_1-b_3)\,4\,\frac{\pi^2_{b_1 b_2 b_3}}{\pi^2_{b_1 b_2}\pi^2_{b_1 b_3}}\cdot\tfrac{1}{4}\pi^2_{b_1 b_2}\cdot\tfrac{1}{4}\pi^2_{b_1 b_3},$$

which, in the case of the dissection (I), is equal to

$$\tfrac{1}{4}(b_3-b_2)(b_2-b_1)(b_3-b_1)\,\pi^2_{b_1 b_2 b_3};$$

we can therefore write, on the whole,

$$\frac{\vartheta_{123}(u)\,\vartheta^2(u)}{\vartheta_1(u)\,\vartheta_2(u)\,\vartheta_3(u)}$$

$$=\epsilon_{123}\sqrt{(b_3-b_2)(b_2-b_1)(b_3-b_1)}\,\tfrac{1}{2}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(x_i-b_2)(x_i-b_3)\,F'(x_i)},$$

wherein the $\pm$ sign denoted by ϵ_{123} is perfectly definite, it being remembered that in $\vartheta_{123}(u)$ the characteristic is the sum, *without reduction*, of the characteristics in $\vartheta_1(u)$, $\vartheta_2(u)$, $\vartheta_3(u)$.

We now proceed to show that

$$\epsilon_{123}=-\epsilon_{23}\,\epsilon_{31}\,\epsilon_{12},$$

where $\epsilon_{12}=\epsilon_{21}$, $\epsilon_{31}=\epsilon_{13}$, $\epsilon_{32}=\epsilon_{23}$ are the signs before introduced.

For denoting $\pi_{b_1 b_2}$ as $\pi_{12}(u)$, and remembering that it is a periodic function, we have found that

$$\pi_{23}(v) = \pi_{23}(u + u^{a, b_1}) = -4\,\frac{\pi_{123}(u)}{\pi_{12}(u)\,\pi_{13}(u)},$$

while also, in case of the dissection (I),

$$\begin{aligned}\pi_{23}(v) &= \frac{2\varepsilon_{23}}{\sqrt{b_3 - b_2}}\,\frac{\vartheta_{23}(v)\,\vartheta(v)}{\vartheta_2(v)\,\vartheta_3(v)} = \frac{2\varepsilon_{23}}{\sqrt{b_3 - b_2}}\,\frac{\vartheta_{123}(u)\,\vartheta_1(u)}{\vartheta_{12}(u)\,\vartheta_{13}(u)} \\ &= \frac{2\varepsilon_{23}}{\sqrt{b_3 - b_2}}\,\frac{\frac{1}{2}\varepsilon_{123}\,\pi_{123}(u)\sqrt{(b_3 - b_2)(b_2 - b_1)(b_3 - b_1)}}{\frac{1}{2}\varepsilon_{12}\,\pi_{12}(u)\sqrt{b_2 - b_1}\,\frac{1}{2}\varepsilon_{13}\,\pi_{13}(u)\,.\sqrt{b_3 - b_1}} \\ &= \frac{4\varepsilon_{23}\,\varepsilon_{123}}{\varepsilon_{12}\,\varepsilon_{13}}\,\frac{\pi_{123}(u)}{\pi_{12}(u)\,\pi_{13}(u)},\end{aligned}$$

and comparing this with the former result we have the equation in question.

The obvious equations

$$\pi_{13} - \pi_{23} = (b_1 - b_2)\,\pi_{123},$$
$$(b_2 - b_3)\,\pi_{23} + (b_3 - b_1)\,\pi_{31} + (b_1 - b_2)\,\pi_{12} = 0,$$

give

$$-(b_2 - b_3)(b_3 - b_1)(b_1 - b_2)\,\pi_{123}^2 = (b_2 - b_3)\,\pi_{23}^2 + (b_3 - b_1)\,\pi_{31}^2 + (b_1 - b_2)\,\pi_{12}^2,$$

and therefore, for the dissections (I), (II), (III), (IV),

$$\begin{aligned}&-\binom{B_2}{B_3}\binom{B_3}{B_1}\binom{B_1}{B_2}\vartheta^2(u)\,\vartheta_{123}^2(u) \\ &\qquad = \binom{B_2}{B_3}\vartheta_1^2(u)\,\vartheta_{23}^2(u) + \binom{B_3}{B_1}\vartheta_{31}^2(u)\,\vartheta_2^2(u) + \binom{B_1}{B_2}\vartheta_{12}^2(u)\,\vartheta_3^2(u);\end{aligned}$$

in particular, for the dissection (I) we have

$$\vartheta^2(u)\,\vartheta_{123}^2(u) = \vartheta_1^2(u)\,\vartheta_{23}^2(u) - \vartheta_2^2(u)\,\vartheta_{31}^2(u) + \vartheta_3^2(u)\,\vartheta_{12}^2(u),$$

an equation which, for our purpose, is of importance.

The corresponding expressions for functions of four suffixes may be set down here. We have

$$\pi_{1234} = \sum_{r=1}^{p}\sum_{s=1}^{p}\frac{y_r\,y_s}{\phi(x_r)\,\phi(x_s)}\,\frac{(x_r - x_s)^2}{F'(x_r)\,F'(x_s)},$$

and hence we find

$$\pi_{12}\,\pi_{34} - \pi_{13}\,\pi_{24} = (b_2 - b_3)(b_1 - b_4)\,\pi_{1234},$$

and

$$(b_2 - b_3)(b_1 - b_4)\,\pi_{23}\,\pi_{14} + (b_3 - b_1)(b_2 - b_4)\,\pi_{31}\,\pi_{24} + (b_1 - b_2)(b_3 - b_4)\,\pi_{12}\,\pi_{34} = 0;$$

thence we deduce

$$(b_2-b_3)(b_3-b_1)(b_1-b_2)(b_1-b_4)(b_2-b_4)(b_3-b_4)\,\pi^2_{1234}$$
$$=(b_2-b_3)(b_1-b_4)\,\pi^2_{23}\pi^2_{14}+(b_3-b_1)(b_2-b_4)\,\pi^2_{31}\pi^2_{24}+(b_1-b_2)(b_3-b_4)\,\pi^2_{12}\pi^2_{34}.$$

SECTION VIII.

The first terms in the expansion of a function of one, two or three suffixes.

Putting

$$\vartheta(u\,|\,u^{a,\,a_r})=\vartheta(0)[A_ru_1+\ldots\ldots+P_ru_p+(u_1,\ldots\ldots,u_p)_2+(u_1,\ldots\ldots,u_p)_3+\ldots\ldots],$$

where $(u_1,\ldots\ldots,u_p)_k$ denotes an integral polynomial in $u_1,\ldots\ldots,u_p$ of order k, we have from the results of Section VI, taking $x_1,\ldots\ldots,x_p$ near to $a_1,\ldots\ldots,a_p$ respectively, and $x_k=a_k+t_k^2$, the equation

$$A_ru_1+\ldots\ldots+P_ru_p=\frac{(-1)^{p-r}\lambda_r i^p t_r}{(-1)^{p-r}i^{p-1}}=i\lambda_r t_r$$
$$=i\lambda_r V_r$$
$$=\frac{1}{\lambda_r}\left[u_p+u_{p-1}\chi_1(a_r)+\ldots\ldots+u_1\chi_{p-1}(a_r)\right];$$

hence the first terms in the expansion of

$$\lambda_r\frac{\vartheta(u\,|\,u^{a,\,a_r})}{\vartheta(0)},$$

that is, the linear terms, are

$$u_p+u_{p-1}\chi_1(a_r)+\ldots\ldots+u_1\chi_{p-1}(a_r);$$

this may be symbolically denoted by

$$\frac{P(\xi)}{\xi-a_r},$$

where, after the division of $P(\xi)$ by $\xi-a_r$, we are to replace ξ^{i-1} by u_i and $\xi^{(0)}$ by u_1.

Consider next

$$\frac{\vartheta(u\,|\,u^{a,\,b_1}+u^{a,\,b_2})}{\vartheta(u)}$$
$$=s_{b_1,b_2}\sqrt{b_2-b_1}\,\frac{\vartheta(u\,|\,u^{a,\,b_1})}{\vartheta(u)}\cdot\frac{\vartheta(u\,|\,u^{a,\,b_2})}{\vartheta(u)}\cdot\tfrac{1}{2}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(x_i-b_2)F'(x_i)};$$

and suppose that b_1, b_2 are taken from among the p branch places $a_1, \ldots, a_p$ and are in ascending order, it will be sufficient to consider the case when they are respectively a_1 and a_2; then denoting $\varepsilon_{a_1 a_2}$ by ε_{12} and supposing $x_k = a_k + t_k^2$, for $k = 1, \ldots, p$, the quantities $t_1, \ldots, t_p$ being small, we have to the first approximation

$$\frac{\vartheta(u \,|\, u^{a,\,a_1} + u^{a,\,a_2})}{\vartheta(u)} = \tfrac{1}{2}\varepsilon_{12}\sqrt{a_2 - a_1}\,.\, i\lambda_1 t_1 \,.\, i\lambda_2 t_2 \left(\frac{\sqrt{f'(a_1)}}{P'(a_1)} \cdot \frac{1}{t_1} - \frac{\sqrt{f'(a_2)}}{P'(a_2)} \cdot \frac{1}{t_2}\right)\frac{1}{a_1 - a_2}$$

$$= \tfrac{1}{2}\varepsilon_{12}\sqrt{a_2 - a_1}\,.\, i^2\lambda_1\lambda_2 \left(\frac{2}{i\lambda_1^2}\, t_2 - \frac{2}{i\lambda_2^2}\, t_1\right)\frac{1}{a_1 - a_2}$$

$$= i\varepsilon_{12}\sqrt{a_2 - a_1}\,\frac{1}{\lambda_1\lambda_2}\,(\lambda_2^2 V_2 - \lambda_1^2 V_1)\,\frac{1}{a_1 - a_2}$$

$$= -\,\varepsilon_{12}\sqrt{a_2 - a_1}\,\frac{1}{\lambda_1\lambda_2}\Big[u_{p-1}\,\frac{\chi_1(a_1) - \chi_1(a_2)}{a_1 - a_2} + \ldots + u_1\,\frac{\chi_{p-1}(a_1) - \chi_{p-1}(a_2)}{a_1 - a_2}\Big],$$

so that if

$$\psi_{p-i}(a_1, a_2) = \frac{\chi_{p-i}(a_1) - \chi_{p-i}(a_2)}{a_1 - a_2},$$

the expansion of the function

$$-\,\frac{\varepsilon_{12}\lambda_1\lambda_2}{\sqrt{a_2 - a_1}}\;\frac{\vartheta(u \,|\, u^{a,\,a_1} + u^{a,\,a_2})}{\vartheta(0)}$$

begins with

$$u_{p-1} + u_{p-2}\psi_2(a_1, a_2) + \ldots + u_1\psi_{p-1}(a_1, a_2);$$

this expression may symbolically, as just explained, be denoted by

$$\left[\frac{P(\xi)}{\xi - a_1} - \frac{P(\xi)}{\xi - a_2}\right]\frac{1}{a_1 - a_2},$$

that is, by

$$P(\xi)/(\xi - a_1)(\xi - a_2),$$

where, after division, we are to replace ξ^{i-1} by u_i and $\xi^{(0)}$ by u_1. It is to be borne in mind that in this statement the characteristic denoted by $u^{a,\,a_1} + u^{a,\,a_2}$ is *unreduced*.

Putting now

$$\zeta_{23} = \varepsilon_{23}\sqrt{a_3 - a_2},\ \zeta_{31} = \varepsilon_{31}\sqrt{a_3 - a_1},\ \zeta_{12} = \varepsilon_{12}\sqrt{a_2 - a_1},$$

where a_1, a_2, a_3 are in ascending order and chosen from among the p branch places $a_1, \ldots, a_p$, and $\varepsilon_{23} = \varepsilon_{a_2 a_3}$, etc., as before, consider the equation

$$\frac{\vartheta(u \mid u^{a, a_1} + u^{a, a_2} + u^{a, a_3})}{\vartheta(u)}$$
$$= -\tfrac{1}{2}\zeta_{23}\zeta_{31}\zeta_{12}\frac{\vartheta(u \mid u^{a, a_1})}{\vartheta(u)} \cdot \frac{\vartheta(u \mid u^{a, a_2})}{\vartheta(u)} \cdot \frac{\vartheta(u \mid u^{a, a_3})}{\vartheta(u)} \sum_{i=1}^{p} \frac{y_i / P'(x_i)}{(x_i - a_1)(x_i - a_2)(x_i - a_3)},$$

which has already been established (Section VII). Writing for the moment

$$\frac{\vartheta(u \mid u^{a, a_1})}{\vartheta(u)} = q_1, \quad \frac{\vartheta(u \mid u^{a, a_2})}{\vartheta(u)} = q_2, \text{ etc.}, \quad \frac{\vartheta(u \mid u^{a, a_1} + u^{a, a_2})}{\vartheta(u)} = q_{12},$$

we have

$$q_1 q_2 q_3 \sum_{i=1}^{p} \frac{y_i}{(x_i - a_1)(x_i - a_2)(x_i - a_3) P'(x_i)}$$
$$= \frac{q_1 q_2 q_3}{a_2 - a_3}\left[\sum_{i=1}^{p} \frac{y_i}{(x_i - a_1)(x_i - a_2) P'(x_i)} - \sum_{i=1}^{p} \frac{y_i}{(x_i - a_1)(x_i - a_3) P'(x_i)}\right],$$
$$= 2\frac{q_1 q_2 q_3}{a_2 - a_3}\left[\frac{q_{12}}{q_1 q_2} \cdot \frac{1}{\zeta_{12}} - \frac{q_{13}}{q_1 q_3}\,\frac{1}{\zeta_{13}}\right],$$
$$= \frac{2}{a_2 - a_3}\left[\frac{q_3 q_{12}}{\zeta_{12}} - \frac{q_2 q_{13}}{\zeta_{13}}\right],$$

and, for small values of the arguments the first terms of this expression, as follows by what has just previously been established, are those symbolically denoted by

$$-\frac{2}{a_2 - a_3}\,\frac{1}{\lambda_1\lambda_2\lambda_3}\left[\frac{P(\xi)}{\xi - a_3} \cdot \frac{P(\xi')}{(\xi' - a_1)(\xi' - a_2)} - \frac{P(\xi')}{\xi' - a_2}\,\frac{P(\xi)}{(\xi - a_1)(\xi - a_3)}\right],$$

where ξ' is a symbol equivalent, in its interpretation, with ξ; hence, if we put

$$\phi(x) = \frac{P(x)}{(x - a_1)(x - a_2)(x - a_3)},$$

the first terms in $\vartheta(u \mid u^{a, a_1} + u^{a, a_2} + u^{a, a_3}) / \vartheta(u)$ are those symbolically denoted by

$$\frac{\zeta_{23}\zeta_{31}\zeta_{12}}{\lambda_1\lambda_2\lambda_3}\phi(\xi)\phi(\xi')[(\xi' - a_3)(\xi - a_1)(\xi - a_2) - (\xi - a_2)(\xi' - a_1)(\xi' - a_3)]/(a_2 - a_3);$$

now, so far as the interpretation is concerned,

$$\begin{aligned}
&(\xi'-a_3)(\xi-a_1)(\xi-a_2)-(\xi-a_2)(\xi'-a_1)(\xi'-a_3)\\
&=(\xi-a_3)(\xi'-a_1)(\xi'-a_2)-(\xi'-a_2)(\xi-a_1)(\xi-a_3)\\
&=\tfrac{1}{2}\left[(\xi-a_1)\{(\xi'-a_3)(\xi-a_2)-(\xi'-a_2)(\xi-a_3)\}\right.\\
&\qquad\left.-(\xi'-a_1)\{(\xi-a_2)(\xi'-a_3)-(\xi-a_3)(\xi'-a_2)\}\right]\\
&=\tfrac{1}{2}\left[(\xi-a_1)(\xi-\xi')(a_2-a_3)-(\xi'-a_1)(\xi-\xi')(a_2-a_3)\right]\\
&=\tfrac{1}{2}(\xi-\xi')^2(a_2-a_3).
\end{aligned}$$

Thus, finally, we may state the result by saying that the function

$$+\frac{\lambda_1\lambda_2\lambda_3}{\zeta_{23}\zeta_{31}\zeta_{12}}\,\frac{\vartheta(u\,|\,u^{a,a_1}+u^{a,a_2}+u^{a,a_3})}{\vartheta(0)}$$

has for the lowest terms in its expansion those of the second degree in $u_1,\ldots,u_p$, which are symbolically denoted by

$$+\tfrac{1}{2}\phi(\xi)\phi(\xi')(\xi-\xi')^2.$$

Thus, for example, when $p=3$, $\phi(\xi)=1$, and the function begins with the terms denoted by $+\frac{1}{2}(\xi-\xi')^2$, namely,

$$+\tfrac{1}{2}(u_3u_1-2u_2^2+u_1u_3)=u_1u_3-u_2^2;$$

while, for $p=4$, $\phi(\xi)=\xi-a_4$; and the function begins with

$$\begin{aligned}
&+\tfrac{1}{2}[\xi\xi'-(\xi+\xi')a_4+a_4^2][\xi^2-2\xi\xi'+\xi'^2],\\
&=-\xi^2\xi'^2+\tfrac{1}{2}\xi^3\xi'+\tfrac{1}{2}\xi\xi'^3\\
&\quad-\tfrac{1}{2}a_4(\xi^3+\xi'^3-\xi\xi'^2-\xi'\xi^2)\\
&\quad-a_4^2(\xi\xi'-\tfrac{1}{2}\xi^2-\tfrac{1}{2}\xi'^2)\\
&=-u_3^2+u_2u_4+a_4(u_2u_3-u_1u_4)+a_4^2(u_1u_3-u_2^2).
\end{aligned}$$

Cf. Schottky, "Abriss einer Theorie der Abelschen Functionen von drei Variabeln" (Leipzig, 1880), p. 149, and Burkhardt, Math. Annal., XXXII (1888), p. 442. The form $\phi(x)=P(x)/(x-a_1)\ldots(x-a_k)$ is in fact that denoted, by Klein, by $\phi^{p+1-2\mu}$, associated with $\vartheta(u\,|\,u^{a,a_1}+\ldots+u^{a,a_k})$.

The results obtained in this section may be summarized as follows: Let

$$\begin{aligned}
\sigma_r(u)&=\lambda_r\,\frac{\vartheta(u\,|\,u^{a,a_r})}{\vartheta(0)},\\
\sigma_{r,s}(u)&=-\frac{\lambda_r\lambda_s}{\zeta_{rs}}\,\frac{\vartheta(u\,|\,u^{a,a_r}+u^{a,a_s})}{\vartheta(0)},\\
\sigma_{r,s,t}(u)&=\frac{\lambda_r\lambda_s\lambda_t}{\zeta_{rs}\zeta_{rt}\zeta_{st}}\,\frac{\vartheta(u\,|\,u^{a,a_r}+u^{a,a_s}+u^{a,a_t})}{\vartheta(0)},
\end{aligned}$$

where the characteristics are *unreduced;* then the first terms in the expansions of these functions are, respectively, those denoted by

$$P(\xi)/(\xi-a_r),\quad P(\xi)/(\xi-a_r)(\xi-a_s),\quad \tfrac{1}{2}\phi(\xi)\phi(\xi')(\xi-\xi')^2,$$

where

$$\phi(\xi)=P(\xi)/(\xi-a_r)(\xi-a_s)(\xi-a_t),\quad \xi^{i-1}=u_i,\quad \xi^{(0)}=u_1.$$

SECTION IX.

Expression of the function $\vartheta(u\,|\,u^{a,\,b_1}+\ldots\ldots+u^{a,\,b_k})$ *in terms of functions* $\vartheta(u),\ \vartheta(u\,|\,u^{a,\,b_r}),\ \vartheta(u\,|\,u^{a,\,b_r}+u^{a,\,b_s})$.

We have, for the dissection (I), and when the characteristic denoted by $u^{a,\,b_1}+u^{a,\,b_2}$ is unreduced, and the places $b_1, b_2, b_3, \ldots\ldots$ are in ascending order,

$$\frac{\vartheta(u\,|\,u^{a,\,b_1}+u^{a,\,b_2})\,\vartheta(u)}{\vartheta(u\,|\,u^{a,\,b_1})\,\vartheta(u\,|\,u^{a,\,b_2})}=\varepsilon_{12}\sqrt{b_2-b_1}\,\tfrac{1}{2}\pi_{12}=\zeta_{12}\,\tfrac{1}{2}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(x_i-b_2)\,F'(x_i)};$$

we write

$$\phi_{12}(u)=\frac{\vartheta(u\,|\,u^{a,\,b_1}+u^{a,\,b_2})}{\zeta_{12}},$$

the characteristic being unreduced. Further we write, for the present,

$$\vartheta_1(u)=\vartheta(u\,|\,u^{a,\,b_1}),\quad \vartheta_{12}=\vartheta(u\,|\,u^{a,\,b_1}+u^{a,\,b_2}),\ \text{etc.}$$

Then we have found

$$\frac{\vartheta_{123}(u)\,\vartheta^2(u)}{\vartheta_1(u)\,\vartheta_2(u)\,\vartheta_3(u)}=-\zeta_{23}\,\zeta_{13}\,\zeta_{12}\,\tfrac{1}{2}\pi_{123},$$

and

$$\pi_{13}-\pi_{23}=(b_1-b_2)\,\pi_{123},$$

so that

$$\frac{1}{\zeta_{13}}\,\frac{\vartheta_{13}(u)\,\vartheta(u)}{\vartheta_1(u)\,\vartheta_3(u)}-\frac{1}{\zeta_{23}}\,\frac{\vartheta_{23}(u)\,\vartheta(u)}{\vartheta_2(u)\,\vartheta_3(u)}=-\frac{b_1-b_2}{\zeta_{23}\,\zeta_{31}\,\zeta_{12}}\,\frac{\vartheta_{123}(u)\,\vartheta^2(u)}{\vartheta_1(u)\,\vartheta_2(u)\,\vartheta_3(u)},$$

or

$$\vartheta(u)\,\vartheta_{123}(u)=-\frac{\zeta_{23}\,\zeta_{31}\,\zeta_{12}}{b_1-b_2}\left[\vartheta_2(u)\,\phi_{13}(u)-\vartheta_1(u)\,\phi_{23}(u)\right],$$

$$=\zeta_{23}\,\zeta_{31}\,\zeta_{12}\begin{vmatrix}\vartheta_1(u) & \vartheta_2(u)\\ \phi_{13}(u) & \phi_{23}(u)\end{vmatrix}\div(b_1-b_2),$$

and it is manifest that the function

$$\frac{\vartheta_1(u)\,\phi_{23}(u)-\vartheta_2(u)\,\phi_{13}(u)}{b_1-b_2},$$

being equal to $\vartheta(u)\,\vartheta_{123}(u)/\zeta_{23}\zeta_{31}\zeta_{12}$, is unaltered by the interchange of any two of the suffixes 1, 2, 3. Further (B. 285), if $a^{a,\,b_2}=\binom{p'}{p}$, $u^{a,\,b_1}+u^{a,\,b_2}=\binom{q'}{q}$,

$$\begin{aligned}\phi_{12}(u+u^{a,\,b_2}) &= e^{\lambda_p(u)-2\pi i p'_q}\frac{\vartheta_{123}(u)}{\zeta_{12}}\\ &= e^{\lambda_p(u)-2\pi i p'_q}\zeta_{13}\zeta_{23}\frac{\vartheta_1(u)\,\phi_{23}(u)-\vartheta_2(u)\,\phi_{13}(u)}{\vartheta(u)\,.\,(b_1-b_2)};\end{aligned}$$

we make frequent use of this fact in what immediately follows; for the sake of brevity the right hand will be denoted, more shortly, by

$$e^{\lambda_p(u)-2\pi i p'_q}\zeta_{13}\zeta_{23}\frac{\vartheta_1\phi_{23}-\vartheta_2\phi_{13}}{\vartheta\,.\,(b_1-b_2)}.$$

Also we put

$$\nabla_{12\,\ldots\,k}=\zeta_{12}\zeta_{13}\zeta_{23}\zeta_{14}\zeta_{24}\zeta_{34}\ldots\zeta_{1k}\zeta_{2k}\ldots\zeta_{k-1,\,k},$$

where any symbol $\zeta_{rs}=\varepsilon_{rs}\,.\,\sqrt{b_r-b_s}$, the difference being taken so that b_s, b_r are in ascending order.

Consider now $2n$ indices $1, 2, 3, \ldots, 2n$; divide them into two sets $r_1, \ldots, r_n$ and $s_1, s_2, \ldots, s_n$; let

$$\begin{aligned}D_n &= (b_{r_1}-b_{r_2})\ldots(b_{r_1}-b_{r_n})(b_{r_2}-b_{r_3})\ldots(b_{r_2}-b_{r_n})\ldots(b_{r_{n-1}}-b_{r_n}),\\ E_n &= (b_{s_1}-b_{s_2})\ldots(b_{s_1}-b_{s_n})(b_{s_2}-b_{s_3})\ldots(b_{s_2}-b_{s_n})\ldots(b_{s_{n-1}}-b_{s_n});\end{aligned}$$

then we proceed to show that

$$\vartheta^{(n-1)}(u)\,\vartheta_{12\,\ldots\,(2n)}(u)=\begin{vmatrix}\phi_{r_1s_1} & \phi_{r_1s_2} & \ldots & \phi_{r_1s_n}\\ \phi_{r_2s_1} & \phi_{r_2s_2} & \ldots & \phi_{r_2s_n}\\ \ldots & \ldots & \ldots & \ldots\\ \phi_{r_ns_1} & \phi_{r_ns_2} & \ldots & \phi_{r_ns_n}\end{vmatrix}\frac{\nabla_{12\,\ldots\,(2n)}}{D_nE_n};$$

and further, taking $(2n+1)$ indices $1, 2, 3, \ldots, (2n+1)$, and dividing them into two sets $r_1, \ldots, r_n$ and $s_1, s_2, \ldots, s_{n+1}$, and putting

$$D_n=(b_{r_1}-b_{r_2})\ldots(b_{r_1}-b_{r_n})\ldots(b_{r_{n-1}}-b_{r_n}),$$

as before, and

$$E_{n+1} = (b_{s_1} - b_{s_2}) \ldots (b_{s_1} - b_{s_{n+1}})(b_{s_2} - b_{s_3}) \ldots \times (b_{s_{n-1}} - b_{s_n})(b_{s_{n-1}} - b_{s_{n+1}})(b_{s_n} - b_{s_{n+1}})$$
$$= E_n (b_{s_1} - b_{s_{n+1}})(b_{s_2} - b_{s_{n+1}}) \ldots (b_{s_n} - b_{s_{n+1}}),$$

then we proceed to prove that

$$\vartheta^{(n)}(u)\,\vartheta_{123\ldots(2n+1)}(u) = \begin{vmatrix} \vartheta_{s_1} & \vartheta_{s_2} & \ldots & \vartheta_{s_{n+1}} \\ \phi_{r_1 s_1} & \phi_{r_1 s_2} & \ldots & \phi_{r_1 s_{n+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{r_n s_1} & \phi_{r_n s_2} & \ldots & \phi_{r_n s_{n+1}} \end{vmatrix} \frac{\nabla_{12\ldots(2n+1)}}{D_n E_{n+1}};$$

the latter result has been shown to hold for $n = 1$; we deduce that if the latter result holds for any value of n, then the former result holds for the value $n + 1$, and that if the former result holds, the latter result holds; by combining these we have the truth of the theorems in general.

If in the latter result we increase u by $u^{a,\, b_{r_{n+1}}}$, we obtain, dividing out by a certain exponential factor,

$$\vartheta^{(n)}_{r_{n+1}}(u)\,\vartheta_{123\ldots(2n+2)}(u)\,\frac{D_n E_{n+1}}{\nabla_{12\ldots(2n+1)}} =$$
$$\begin{vmatrix} \zeta_{r_{n+1}s_1}\phi_{r_{n+1}s_1}, & \ldots, & \zeta_{r_{n+1}s_{n+1}}\phi_{r_{n+1}s_{n+1}} \\ \zeta_{r_1 r_{n+1}}\zeta_{s_1 r_{n+1}}\dfrac{1}{\vartheta}\dfrac{\vartheta_{r_1}\phi_{r_{n+1}s_1} - \vartheta_{r_{n+1}}\phi_{r_1 s_1}}{b_{r_1} - b_{r_{n+1}}}, & \ldots, & \zeta_{r_1 r_{n+1}}\zeta_{r_{n+1}s_{n+1}}\dfrac{1}{\vartheta}\cdot\dfrac{\vartheta_{r_1}\phi_{r_{n+1}s_{n+1}} - \vartheta_{r_{n+1}}\phi_{r_1 s_{n+1}}}{b_{r_1} - b_{r_{n+1}}} \\ \cdots & \cdots & \cdots \\ \zeta_{r_n r_{n+1}}\zeta_{s_1 r_{n+1}}\dfrac{1}{\vartheta}\dfrac{\vartheta_{r_n}\phi_{r_{n+1}s_1} - \vartheta_{r_{n+1}}\phi_{r_n s_1}}{b_{r_n} - b_{r_{n+1}}}, & \ldots, & \zeta_{r_n r_{n+1}}\zeta_{r_{n+1}s_{n+1}}\dfrac{1}{\vartheta}\cdot\dfrac{\vartheta_{r_n}\phi_{r_{n+1}s_{n+1}} - \vartheta_{r_{n+1}}\phi_{r_n s_{n+1}}}{b_{r_n} - b_{r_{n+1}}} \end{vmatrix}$$

$$= \frac{\zeta_{r_1 r_{n+1}} \ldots \zeta_{r_n r_{n+1}}\zeta_{r_{n+1}s_1} \ldots \zeta_{r_{n+1}s_{n+1}}}{(b_{r_1} - b_{r_{n+1}}) \ldots (b_{r_n} - b_{r_{n+1}})}\,\frac{\vartheta^{(n)}_{r_{n+1}}}{\vartheta^{(n)}} \begin{vmatrix} \phi_{r_1 s_1} & \ldots & \phi_{r_1 s_{n+1}} \\ \phi_{r_2 s_1} & \ldots & \phi_{r_2 s_{n+1}} \\ \cdots & \cdots & \cdots \\ \phi_{r_{n+1}s_1} & \ldots & \phi_{r_{n+1}s_{n+1}} \end{vmatrix}$$

and hence

$$\mathfrak{I}^n(u)\,\mathfrak{I}_{123\ldots(2n+2)}(u) = \begin{vmatrix} \phi_{r_1 s_1} & \ldots \phi_{r_1 s_{n+1}} \\ \phi_{r_2 s_1} & \ldots \phi_{r_2 s_{n+1}} \\ \ldots & \ldots \\ \phi_{r_{n+1} s_1} & \ldots \phi_{r_{n+1} s_{n+1}} \end{vmatrix} \frac{\nabla_{12\ldots(2n+2)}}{D_{n+1}E_{n+1}}.$$

Further, assuming

$$\mathfrak{I}^{(n-1)}(u)\,\mathfrak{I}_{12\ldots(2n)}(u)\,\frac{D_n E_n}{\nabla_{12\ldots(2n)}} = \begin{vmatrix} \phi_{r_1 s_1}, \ldots, \phi_{r_1 s_n} \\ \ldots \\ \phi_{r_n s_1}, \ldots, \phi_{r_n s_n} \end{vmatrix},$$

we obtain, increasing u by $u^{a,\,b_{s_{n+1}}}$, and dividing out by a certain exponential factor,

$$\mathfrak{I}^{(n-1)}_{s_{n+1}}(u)\,\mathfrak{I}_{12\ldots(2n+1)}(u)\,\frac{D_n E_n}{\nabla_{12\ldots(2n)}} =$$

$$\begin{vmatrix} \zeta_{r_1 s_{n+1}}\zeta_{s_1 s_{n+1}}\dfrac{1}{\mathfrak{I}}\dfrac{\mathfrak{I}_{s_1}\phi_{r_1 s_{n+1}} - \mathfrak{I}_{s_{n+1}}\phi_{r_1 s_1}}{b_{s_1} - b_{s_{n+1}}}, \ldots, \zeta_{r_1 s_{n+1}}\zeta_{s_n s_{n+1}}\dfrac{1}{\mathfrak{I}}\dfrac{\mathfrak{I}_{s_n}\phi_{r_1 s_{n+1}} - \mathfrak{I}_{s_{n+1}}\phi_{r_1 s_n}}{b_{s_n} - b_{s_{n+1}}} \\ \ldots\ldots\ldots \\ \zeta_{r_n s_{n+1}}\zeta_{s_1 s_{n+1}}\dfrac{1}{\mathfrak{I}}\dfrac{\mathfrak{I}_{s_1}\phi_{r_n s_{n+1}} - \mathfrak{I}_{s_{n+1}}\phi_{r_n s_1}}{b_{s_1} - b_{s_{n+1}}}, \ldots, \zeta_{r_n s_{n+1}}\zeta_{s_n s_{n+1}}\dfrac{1}{\mathfrak{I}}\dfrac{\mathfrak{I}_{s_n}\phi_{r_n s_{n+1}} - \mathfrak{I}_{s_{n+1}}\phi_{r_n s_n}}{b_{s_n} - b_{s_{n+1}}} \end{vmatrix}$$

$$= (-1)^n \frac{\zeta_{r_1 s_{n+1}}\ldots\zeta_{r_n s_{n+1}}\zeta_{s_1 s_{n+1}}\ldots\zeta_{s_n s_{n+1}}}{(b_{s_1} - b_{s_{n+1}})\ldots(b_{s_n} - b_{s_{n+1}})}\,\frac{1}{\mathfrak{I}^n}$$

$$\begin{vmatrix} \mathfrak{I}_{s_1}, \ldots, \mathfrak{I}_{s_n}, & 1 \\ \mathfrak{I}_{s_1}\phi_{r_1 s_{n+1}} - \mathfrak{I}_{s_{n+1}}\phi_{r_1 s_1}, \ldots, \mathfrak{I}_{s_n}\phi_{r_1 s_{n+1}} - \mathfrak{I}_{s_{n+1}}\phi_{r_1 s_n}, & 0 \\ \ldots\ldots & \ldots \\ \mathfrak{I}_{s_1}\phi_{r_n s_{n+1}} - \mathfrak{I}_{s_{n+1}}\phi_{r_n s_1}, \ldots, \mathfrak{I}_{s_n}\phi_{r_n s_{n+1}} - \mathfrak{I}_{s_{n+1}}\phi_{r_n s_n}, & 0 \end{vmatrix}$$

$$= \frac{\zeta_{r_1 s_{n+1}}\ldots\zeta_{r_n s_{n+1}}\zeta_{s_1 s_{n+1}}\ldots\zeta_{s_n s_{n+1}}}{(b_{s_1} - b_{s_{n+1}})\ldots(b_{s_n} - b_{s_{n+1}})}\,\frac{\mathfrak{I}^{(n-1)}_{s_{n+1}}}{\mathfrak{I}^{(n)}} \begin{vmatrix} \mathfrak{I}_{s_1}, \ldots, \mathfrak{I}_{s_n}, \mathfrak{I}_{s_{n+1}} \\ \phi_{r_1 s_1}, \ldots, \phi_{r_1 s_n}, \phi_{r_1 s_{n+1}} \\ \ldots \\ \phi_{r_n s_1}, \ldots, \phi_{r_n s_n}, \phi_{r_n s_{n+1}} \end{vmatrix},$$

so that

$$\mathfrak{I}^n(u)\,\mathfrak{I}_{12\ldots 2n+1}(u) = \begin{vmatrix} \mathfrak{I}_{s_1}, \ldots, \mathfrak{I}_{s_n}, \mathfrak{I}_{s_{n+1}} \\ \phi_{r_1 s_1}, \ldots, \phi_{r_1 s_n}, \phi_{r_1 s_{n+1}} \\ \ldots \\ \phi_{r_n s_1}, \ldots, \phi_{r_n s_n}, \phi_{r_n s_{n+1}} \end{vmatrix} \frac{\nabla_{12\ldots(2n+1)}}{D_n E_{n+1}}.$$

SUMMARY OF SECTION IX.

These results may be summarized as follows: Let $\lambda_1, \lambda_2, \ldots$ be any quantities whatever; let

$$\vartheta_{123\ldots k}(u) = \vartheta(u \mid u^{a_1 b_1} + \ldots + u^{a_k b_k}),$$

where $b_1, \ldots, b_k$ are any finite branch places, and the characteristic is *unreduced;* let $\zeta_{rs} = \varepsilon_{rs}\sqrt{b_r - b_s}$, where the difference is to be taken so that b_s, b_r are in ascending order; let $\nabla_{1\ldots k}$ be the product of all the $\frac{1}{2}k(k-1)$ quantities ζ_{rs} in which r, s are a pair of different suffixes chosen from $1, \ldots, k$; then putting

$$\sigma(u) = \frac{\vartheta(u)}{\vartheta(0)},$$

$$\sigma_{12\ldots(2n)}(u) = (-1)^n \frac{\lambda_1\lambda_2\ldots\lambda_{2n}}{\nabla_{12\ldots(2n)}} \frac{\vartheta_{12\ldots(2n)}(u)}{\vartheta(0)}, \quad (n = 1, 2\ldots)$$

$$\sigma_{12\ldots(2n+1)}(u) = \frac{\lambda_1\lambda_2\ldots\lambda_{2n+1}}{\nabla_{12\ldots(2n+1)}} \frac{\vartheta_{12\ldots(2n+1)}(u)}{\vartheta(0)}, \quad (n = 0, 1, 2, \ldots)$$

we have

$$\sigma^{(n-1)}(u)\,\sigma_{12\ldots(2n)}(u) = \begin{vmatrix} \sigma_{r_1 s_1}(u), & \ldots, & \sigma_{r_1 s_n}(u) \\ \ldots & \ldots & \ldots \\ \sigma_{r_n s_1}(u), & \ldots, & \sigma_{r_n s_n}(u) \end{vmatrix} \frac{1}{D_n E_n},$$

and

$$\sigma^{(n)}(u)\,\sigma_{12\ldots(2n+1)}(u) = \begin{vmatrix} \sigma_{r_1 s_1}(u), & \ldots, & \sigma_{r_1 s_{n+1}}(u) \\ \ldots & \ldots & \ldots \\ \sigma_{r_n s_1}(u), & \ldots, & \sigma_{r_n s_{n+1}}(u) \\ \sigma_{s_1}(u), & \ldots, & \sigma_{s_{n+1}}(u) \end{vmatrix} \frac{1}{D_n E_{n+1}}$$

where, in the former, if $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ be any decomposition of the indices $1, 2, \ldots, (2n)$ into two sets of n each, then

$$D_n = (b_{r_1} - b_{r_2})\ldots(b_{r_1} - b_{r_n})(b_{r_2} - b_{r_3})\ldots(b_{r_2} - b_{r_n})\ldots(b_{r_{n-1}} - b_{r_n}),$$
$$E_n = (b_{s_1} - b_{s_2})\ldots(b_{s_1} - b_{s_n})(b_{s_2} - b_{s_3})\ldots(b_{s_2} - b_{s_n})\ldots(b_{r_{n-1}} - b_{r_n}),$$

and in the latter, if $r_1, \ldots, r_n$ and $s_1, \ldots, s_{n+1}$ be any decomposition of $1, 2, \ldots, (2n+1)$ into two sets of respectively n and $n+1$ terms, then D_n is as before and

$$E_{n+1} = E_n(b_{s_1} - b_{s_{n+1}})(b_{s_2} - b_{s_{n+1}})\ldots(b_{s_n} - b_{s_{n+1}}).$$

In these results, as already stated, $b_1, \ldots, b_{2n+1}$ are any finite branch places, and, correspondingly, $\lambda_1, \ldots, \lambda_{2n+1}$ are arbitrary quantities.

But in the application which we make of the formulæ, $b_1, \ldots, b_{2n+1}$ *will be chosen only from the p branch places* $a_1, \ldots, a_p$, *and, correspondingly,* $\lambda_1, \ldots, \lambda_{2n+1}$ *will have the values before assigned to them* (*Section VI*). It follows then from the form of these equations that, in the expansion of the σ functions in powers of $u_1, \ldots, u_p$, the function $\sigma_{12\ldots(2n)}(u)$ will begin with terms of degree n, and the function $\sigma_{12\ldots(2n+1)}(u)$ will begin with terms of degree $(n+1)$. In other words, $\sigma_{12\ldots k}(u)$ *vanishes to order* $\frac{1}{2}k$ *or* $\frac{1}{2}(k+1)$ *when the arguments* $u_1, \ldots, u_p$ *vanish, and has parity* $(-1)^{\frac{1}{2}k}$, *or* $(-1)^{\frac{1}{2}(k+1)}$ *according as k is even or odd.* We shall obtain presently the actual forms of the leading terms in the expansions of the functions, from which it will appear that the functions agree with those considered by Klein. In fact, for $k = 2n$ the function $P(x)/(x-a_1)\ldots(x-a_k)$ is to be identified with that denoted by Klein by $\phi_x^{p+1-2\mu}$, the additional factor being $x-a$ which is here to be replaced by 1; the $\frac{1}{2}k$ is to be identified with the symbol μ used by Klein; in other words, the function $\sigma_{12\ldots(2n)}(u)$ is that whose algebraical characteristic is given by the decomposition

$$(x-a)\,P(x)/(x-a_1)\ldots(x-a_k);\quad Q(x).(x-a_1)\ldots(x-a_k).$$

Similarly for $k = 2n+1$, we have $\mu = \frac{1}{2}(k+1)$, and $\sigma_{12\ldots(2n+1)}(u)$ corresponds to the decomposition

$$P(x)/(x-a_1)\ldots(x-a_k);\quad (x-a)\,Q(x)(x-a_1)\ldots(x-a_k).$$

(Compare B. 436).

We now obtain the leading terms in the expansions of each of the functions $\sigma_{12\ldots(2n)}(u)$, $\sigma_{12\ldots(2n+1)}(u)$. The deduction is an immediate consequence of the fact (Scott, "Determinants," Cambridge, 1880, p. 121) that the determinant

$$\begin{vmatrix} \dfrac{1}{x_1-a_1}, & \dfrac{1}{x_1-a_2}, & \ldots, & \dfrac{1}{x_1-a_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{1}{x_n-a_1}, & \dfrac{1}{x_n-a_2}, & \ldots, & \dfrac{1}{x_n-a_n} \end{vmatrix}$$

is equal to

$$(-1)^{\frac{1}{2}n(n-1)}\prod_{i<j}\prod(x_i-x_j)\prod_{i<j}\prod(a_i-a_j)/\prod_{i=1}^{n}(x_i-a_1)\ldots(x_i-a_n).$$

From this we find (cf. Section VIII, Summary) that if $\xi_1, \ldots, \xi_n$ be equivalent symbols, to be afterwards interpreted by the rule

$$\xi_i^{r-1} = u_r, \quad \xi_i^{(0)} = u_1, \qquad (i = 1, \ldots, n; \; r = 2, \ldots, p)$$

and $\phi(x) = P(x)/(x - a_1) \ldots (x - a_{2n})$, then the function $\sigma_{12 \ldots (2n)}(u)$ begins with the terms denoted by

$$\frac{1}{n!} \phi(\xi_1) \ldots \phi(\xi_n) \Delta(\xi_1, \ldots, \xi_n),$$

where $\Delta(\xi_1, \ldots, \xi_n)$ is the product of the squares of the differences of the quantities $\xi_1, \ldots, \xi_n$. While similarly the function $\sigma_{12 \ldots (2n+1)}(u)$ begins with the terms denoted by

$$\frac{1}{(n+1)!} \phi(\xi_1) \ldots \phi(\xi_{n+1}) \Delta(\xi_1, \ldots, \xi_{n+1}),$$

where $\phi(x) = P(x)/(x - a_1) \ldots (x - a_{2n+1})$. Cf. Burkhardt, Math. Annal., XXXII (1888), p. 442.

SECTION X.

Expression of the square of a theta function of three or more suffixes in terms of the squares of the functions of one and two suffixes.

The results of the present section are conveniently representable in terms of the algebraical expressions known as Pfaffians, in regard to which, therefore, some preliminary remarks should be made (cf. e. g. Scott, "Determinants," Cambridge, 1880, p. 74).

Let (12) denote any algebraical quantity, and $(21) = -(12)$, and let $(12 \ldots k)$ denote the integral rational expression, of order k in the quantities (rs), which is defined by the equation

$$\begin{aligned}(12 \ldots k) &= (12)(34 \ldots k) - (13)(245) \ldots k) + (14)(235 \ldots k) \\ &\qquad - \ldots + (1k)(234 \ldots \overline{k-1}) \\ &= (12)(34) \ldots (\overline{k-1k}) \pm \ldots,\end{aligned}$$

where the other terms of the latter form are obtainable from the one written down by interchange of the indices 2, 3, 4, ..., with proper changes of sign. We suppose k to be even, $= 2n$; when k is odd the function vanishes identi-

cally. The square of the function is equal to the determinant whose $(i, j)^{\text{th}}$ element is (ij) when $i<j$, and $-(ji)$ when $i>j$.

In regard to this expression we utilize the two results:

(I). If in the expanded result every element such as (12) be replaced by $(xy12)$, that is, by

$$(xy)(12)-(x1)(y2)+(x2)(y1),$$

x and y being indices other than 1, 2, ..., $2n$, the result will be

$$(xy)^{n-1}(xy1234\ldots\ldots\overline{2n-1}\,.\,2n),$$

namely, we have

$$(xy12)(xy34)\ldots(xy\overline{2n-1}\,.\,2n)\pm\ldots=(xy)^{n-1}(xy1234\ldots\overline{2n-1}\,.\,2n). \quad \text{(I)}$$

For instance, n being 2, it is easily verified by direct calculation that

$$(xy12)(xy34)-(xy13)(xy24)+(xy14)(xy23)=(xy)(xy1234).$$

(II). If in the expanded expression of the Pfaffian $(y123\ldots\ldots\overline{2n+1})$, namely,

$$(y1)(23)(45)\ldots\ldots(2n\,.\,\overline{2n+1})\pm\ldots\ldots,$$

we replace every element whose symbol does not contain the index y, such as (23), by a Pfaffian of four indices $(xy23)$, and replace every element involving y, such as $(y1)$, by the corresponding element in which x replaces y, namely $(x1)$, the result will be

$$(xy)^n(x123\ldots\ldots\overline{2n+1}),$$

namely, we have

$$(x1)(xy23)(xy45)\ldots\ldots(xy2n\,.\,\overline{2n+1})\pm\ldots\ldots=(xy)^n(x123\ldots\ldots\overline{2n+1}). \quad \text{(II)}$$

For instance, when $n=1$, we have, as is easily verified,

$$(x1)(xy23)-(x2)(xy13)+(x3)(xy12)=(xy)(x123).$$

To prove (I), notice that the expansion

$$[(xy)(12)-(x1)(y2)+(x2)(y1)]\ldots\ldots[(xy)(x\beta)-(x\alpha)(y\beta)+(x\beta)(y\alpha)]\ldots\ldots$$
$$[(xy)(2n-1\,.\,2n)-(x2n-1)(y2n)+(x2n)(y2n-1)]\pm\ldots\ldots$$

is

$$\begin{aligned}&(xy)^n(123\ldots\ldots2n)\\&+(xy)^{n-1}(12)(34)\ldots\ldots(2n-1\ 2n)(x\beta)(y\alpha)\pm\ldots\ldots\\&+(xy)^{n-2}(12)(34)\ldots\ldots(2n-1\ 2n)(x\beta)(y\alpha)(x\beta')(y\alpha')\pm\ldots\ldots\\&+\ldots\ldots\\&+(x\beta_1)(x\beta_2)\ldots\ldots(x\beta_n)(y\alpha_1)\ldots\ldots(y\alpha_n)\pm\ldots\ldots,\end{aligned}$$

wherein, in the second line, the product $(12)(34)\ldots(2n-1\;2n)$ does not contain the factor $(\alpha\beta)$, and in the third line the product $(12)(34)\ldots(2n-1\;2n)$ does not contain the terms $(\alpha\beta)$, $(\alpha'\beta')$, and so on.

Now the aggregate of the terms in the last line, which are those in which no factor (xy) enters, is zero; for to every term $(x\beta_1)\ldots(x\beta_n)(y\alpha_1)\ldots(y\alpha_n)$ we may make correspond another term, also occurring in this line, which differs from this one only in the interchange of two of the indices $\beta_1,\ldots,\beta_n$; on account of this single interchange this corresponding term will have a different sign from that of the term written down, and the two terms will cancel one another.

A similar argument applies to every line of the expansion except the first and second; for instance, in the third line corresponding to the term

$$(xy)^{n-2}(12)(34)\ldots(2n-1\;2n)(x\beta)(y\alpha)(x\beta')(y\alpha')$$

there is a term

$$(xy)^{n-2}(12)(34)\ldots(2n-1\;2n)(x\beta')(y\alpha)(x\beta)(y\alpha'),$$

which, on account of the interchange of β and β', has a different sign from that of the term written down.

The first and second lines together clearly give

$$\begin{aligned}&(xy)^{n-1}\left[(xy)(123\ldots2n)-(x1)(y23\ldots2n)+(x2)(y13\ldots2n)-\ldots\right]\\ &=(xy)^{n-1}(xy12\ldots2n),\end{aligned}$$

which proves the theorem.

The result (II) can be deduced from (I). In virtue of that theorem, the expression on the left side of equation (II) is in fact

$$\begin{aligned}&(x1)(xy)^{n-1}(xy23\ldots2n+1)-(x2)(xy)^{n-1}(xy13\ldots2n+1)+\ldots\\ &\qquad\qquad+(x2n+1)(xy)^{n-1}(xy12\ldots2n)\\ &=(xy)^{n-1}\left[(x1)(xy23\ldots2n+1)-(x2)(xy13\ldots2n+1)+\ldots\right.\\ &\qquad\qquad\left.+(x2n+1)(xy12\ldots2n)\right]\\ &=(xy)^{n-1}\left[(xxy12\ldots2n+1)+(xy)(x123\ldots2n+1)\right]\\ &=(xy)^{n}(x123\ldots2n+1),\ \text{since}\ (xxy12\ldots2n+1)=0;\end{aligned}$$

and this proves the result in question.

In the applications now to be made of these theorems (I) and (II), we put $\vartheta_{rs}(u)$ to denote $\vartheta(u\,|\,u^{a,b_r}+u^{a,b_s})$, where b_r, b_s are any two finite branch

places, and, more generally, $\vartheta_{12\ldots k}(u)$ to denote $\vartheta(u|u^{a_1,b_1}+\ldots+u^{a_k,b_k})$, and we put, for the elements (rs) of the Pfaffians which occur,

$$\begin{aligned} \text{when } r<s,\ (rs) &= \quad \vartheta^2_{rs}(u), \text{ with } (0s) = \quad \vartheta^2_s(u), \\ r>s,\ (rs) &= -\vartheta^2_{rs}(u), \text{ with } (r0) = -\vartheta^2_r(u); \end{aligned}$$

for the sake of brevity we replace $\vartheta^2_{rs}(u)$ simply by ϑ^2_{rs}, and $\vartheta^2_r(u)$ by ϑ^2_r.

The results to be obtained are those represented by the equations

$$\left.\begin{aligned} \vartheta^{2(n-1)}(u)\,\vartheta^2_{12\ldots 2n}(u) &= (12\ldots 2n), \\ \vartheta^{2n}(u)\,\vartheta^2_{12\ldots 2n+1}(u) &= (012\ldots 2n+1), \end{aligned}\right\} \quad (n=1,2,3,\ldots) \quad \text{(III)}$$

which hold for the dissection (I) of the Riemann surface by means of which the theta functions are defined.

The method of proof is by induction. We have already proved (Section VII) that

$$\vartheta^2(u)\,\vartheta^2_{123}(u) = \vartheta^2_1(u)\,\vartheta^2_{23}(u) - \vartheta^2_2(u)\,\vartheta^2_{13}(u) + \vartheta^2_3(u)\,\vartheta^2_{12}(u),$$

or, as we write it,

$$\vartheta^2\,\vartheta^2_{123} = (0123);$$

we assume that

$$\vartheta^{2n}\,\vartheta^2_{12\ldots(2n+1)} = (012\ldots 2n\,.\,2n+1),$$

and we deduce that

$$\vartheta^{2n}\,\vartheta^2_{12\ldots(2n+2)} = (123\ldots 2n+1\,.\,2n+2),$$

and

$$\vartheta^{2n+2}\,\vartheta^2_{12\ldots(2n+3)} = (012\ldots 2n+2\,.\,2n+3);$$

these suffice to prove the results (III) in all cases.

It is worth while to illustrate the nature of the proof in detail by deducing from $\vartheta^2\,\vartheta^2_{123} = (0123)$ the next following case, namely, the equation

$$\vartheta^2\,\vartheta^2_{1234} = \vartheta^2_{12}\,\vartheta^2_{34} - \vartheta^2_{13}\,\vartheta^2_{24} + \vartheta^2_{14}\,\vartheta^2_{23}.$$

We have, as cases of the equation $\vartheta^2\,\vartheta^2_{123} = (0123)$, the following, where 1, 2, 3, 4 are associated with any four finite branch places in ascending order, namely,

$$\begin{aligned} \vartheta^2\,\vartheta^2_{234} &= \vartheta^2_2\,\vartheta^2_{34} - \vartheta^2_3\,\vartheta^2_{24} + \vartheta^2_4\,\vartheta^2_{23}, \\ \vartheta^2\,\vartheta^2_{134} &= \vartheta^2_1\,\vartheta^2_{34} - \vartheta^2_3\,\vartheta^2_{14} + \vartheta^2_4\,\vartheta^2_{13}, \\ \vartheta^2\,\vartheta^2_{124} &= \vartheta^2_1\,\vartheta^2_{24} - \vartheta^2_2\,\vartheta^2_{14} + \vartheta^2_4\,\vartheta^2_{12}; \end{aligned}$$

from these we deduce

$$\vartheta^2(\vartheta^2_{14}\vartheta^2_{234}-\vartheta^2_{24}\vartheta^2_{134}+\vartheta^2_{34}\vartheta^2_{124})=\vartheta^2_4(\vartheta^2_{12}\vartheta^2_{34}-\vartheta^2_{13}\vartheta^2_{24}+\vartheta^2_{14}\vartheta^2_{23});$$

but from $\vartheta^2\vartheta^2_{123}=(0123)$, by increasing the argument by the half-period u^{a,b_4}, we deduce

$$\vartheta^2_4\vartheta^2_{1234}=\vartheta^2_{14}\vartheta^2_{234}-\vartheta^2_{24}\vartheta^2_{134}+\vartheta^2_{34}\vartheta^2_{124},$$

hence the result in question follows. In other words, the functions $\vartheta^2(u)\vartheta^2_{123}(u)$, $\vartheta^2(u)\vartheta^2_{1234}(u)$ are square roots respectively of the determinants

$$\begin{vmatrix} 0, & \vartheta^2_1, & \vartheta^2_2, & \vartheta^2_3 \\ -\vartheta^2_1, & 0, & \vartheta^2_{12}, & \vartheta^2_{13} \\ -\vartheta^2_2, & -\vartheta^2_{12}, & 0, & \vartheta^2_{23} \\ -\vartheta^2_3, & -\vartheta^2_{13}, & -\vartheta^2_{23}, & 0 \end{vmatrix}, \quad \begin{vmatrix} 0, & \vartheta^2_{12}, & \vartheta^2_{13}, & \vartheta^2_{14} \\ -\vartheta^2_{12}, & 0, & \vartheta^2_{23}, & \vartheta^2_{24} \\ -\vartheta^2_{13}, & -\vartheta^2_{23}, & 0, & \vartheta^2_{34} \\ -\vartheta^2_{14}, & -\vartheta^2_{24}, & -\vartheta^2_{34}, & 0 \end{vmatrix}.$$

Similarly, for the next case, we deduce from $\vartheta^2\vartheta^2_{1234}=(1234)$ that

$$\vartheta^2_5\vartheta^2_{12345}=\vartheta^2_{125}\vartheta^2_{345}-\vartheta^2_{135}\vartheta^2_{245}+\vartheta^2_{145}\vartheta^2_{235},$$

and hence, by substitution of the Pfaffian expressions for $\vartheta^2\vartheta^2_{125}$, etc., that $\vartheta^4\vartheta^2_{12345}=(012345)$, which is a square root of the determinant

$$\begin{vmatrix} 0, & \vartheta^2_1, & \vartheta^2_2, & \vartheta^2_3, & \vartheta^2_4, & \vartheta^2_5 \\ -\vartheta^2_1, & 0, & \vartheta^2_{12}, & \vartheta^2_{13}, & \vartheta^2_{14}, & \vartheta^2_{15} \\ -\vartheta^2_2, & -\vartheta^2_{12}, & 0, & \vartheta^2_{23}, & \vartheta^2_{24}, & \vartheta^2_{25} \\ -\vartheta^2_3, & -\vartheta^2_{13}, & -\vartheta^2_{23}, & 0, & \vartheta^2_{34}, & \vartheta^2_{35} \\ -\vartheta^2_4, & -\vartheta^2_{14}, & -\vartheta^2_{24}, & -\vartheta^2_{34}, & 0, & \vartheta^2_{45} \\ -\vartheta^2_5, & -\vartheta^2_{15}, & -\vartheta^2_{25}, & -\vartheta^2_{35}, & -\vartheta^2_{45}, & 0 \end{vmatrix}.$$

We pass now to the proof of the general formulæ (III); if in the equation $\vartheta^{2n}\vartheta^2_{12\,\ldots\,(2n+1)}=(012\,\ldots\,2n\;2n+1)$, namely,

$$\vartheta^{2n}\vartheta^2_{12\,\ldots\,(2n+1)}=\vartheta^2_1\vartheta^2_{23}\ldots\vartheta^2_{2n\,.\,2n+1}\pm\ldots,$$

we increase the argument by the half-period $u^{a,b_{2n+2}}$, we obtain

$$\vartheta^{2n}\vartheta^{2n}_{2n+2}\vartheta^2_{12\,\ldots\,(2n+2)}=\vartheta^2_{1\,2n+2}\,.\,\vartheta^2\vartheta^2_{23\,2n+2}\ldots\vartheta^2\vartheta^2_{2n\,.\,2n+1\,.\,2n+2}\pm\ldots;$$

here the right side is

$$\begin{aligned}&(1\,.\,2n+2)(023\;2n+2)\ldots(0\,.\,2n\,.\,2n+1\,.\,2n+2)\pm\ldots,\\ &=[(x1)(xy23)(xy45)\ldots(xy\;2n\;2n+1)\pm\ldots](-1)^{n+1},\end{aligned}$$

where x is put for $2n+2$ and y for 0; by the theorem (II) of this section this is equal to

$$\begin{aligned}&(-1)^{n+1}(xy)^n(x123\ldots\ldots 2n+1),\\ &=(0\ 2n+2)^n(123\ldots\ldots 2n+2),\\ &=\vartheta^{2n}_{2n+2}(123\ldots\ldots 2n+2);\end{aligned}$$

thus

$$\vartheta^{2n}\vartheta^2_{12\ldots\ldots(2n+2)}=(123\ldots\ldots 2n+2). \qquad (\alpha)$$

Replacing, in the last obtained equation, n by $m-1$ for convenience of writing, we have

$$\vartheta^{2(m-1)}\vartheta^2_{12\ldots\ldots 2m}=\vartheta^2_{12}\vartheta^2_{34}\ldots\ldots\vartheta^2_{2m-1.2m}\pm\ldots\ldots,$$

and hence, adding a half-period $u^{a,\,b_{2m+1}}$ to the argument,

$$\begin{aligned}\vartheta^{2m}\vartheta^{2(m-1)}_{2m+1}\vartheta^2_{12\ldots\ldots(2m+1)}&=\vartheta^2\vartheta^2_{12.2m+1}\vartheta^2\vartheta^2_{34\,2m+1}\ldots\ldots\vartheta^2\vartheta^2_{2m-1.2m.2m+1}\pm\ldots,\\ &=(012\ 2m+1)(034\ 2m+1)\ldots(0\ 2m-1\ 2m\ 2m+1)\pm\ldots,\\ &=(xy12)(xy34)\ldots(xy\ 2m-1\ 2m)\pm\ldots,\end{aligned}$$

where x, y replace 0 and $2m+1$ respectively; by the theorem (I) of this section the right side is equal to

$$\begin{aligned}&(xy)^{m-1}(xy1234\ldots\ldots 2m-1.2m)\\ &=(0\ 2m+1)^{m-1}(01234\ldots\ldots 2m.2m+1)\\ &=\vartheta^{2(m-1)}_{2m+1}.(012\ldots\ldots 2m.2m+1);\end{aligned}$$

hence, dividing by $\vartheta^{2(m-1)}_{2m+1}$ and replacing m by $n+1$, we have

$$\vartheta^{2(n+1)}\vartheta^2_{12\ldots\ldots(2n+3)}=(012\ldots\ldots 2n+2.2n+3). \qquad (\beta)$$

Now as the first of the equations (III) is identical when $n=1$, and the second of these equations holds for $n=1$, it follows from the equation (α) that the first of the equations (III) holds when $n=2$, and thence, from the equation (β), that the second of the equations (III) holds for $n=2$, and so on.

Two remarks are to be made:

In the first place the two results in equation (III) are not to be regarded as independent; since if in $\vartheta^2_{12\ldots\ldots(2n+1)}(u)$ we suppose the branch place b_{2n+1} to be the branch place a, at infinity, the function becomes the function $\vartheta^2_{12\ldots\ldots 2n}(u)$, while similarly the results for smaller values of n are to be regarded as particular cases of the results for larger values. (Cf. B. 473, 525.)

In the next place, in the applications which are here to be made of the equation (III), it will be supposed that the branch places $b_1, b_2, \ldots\ldots$ are chosen from among the p places $a_1, \ldots\ldots, a_p$.

SECTION XI.

Proof of an addition equation for hyperelliptic theta functions.

The equation in question is given by Königsberger, Crelle, LXIV (see also B. 457). Let B_r denote the characteristic associated with the half-period u^{a, b_r}, b_r being one of the $2p+1$ finite branch places; let A be an arbitrary half integer characteristic. Denote the 2^p characteristics A, $AB_1 \ldots . B_k$, formed by adding the characteristic A to any combination of 0, 1, 2, ..., p of the characteristics $B_1, \ldots, B_p$ by $Q_1, Q_2, \ldots, Q_s$, where $s = 2^p$. Then if C be the characteristic associated with the half-period $u^{b_1, a_1} + \ldots + u^{b_p, a_p}$, every one of the quantities $\vartheta(0 \mid CQ_i Q_j)$, where i is not equal to j, vanishes in the hyperelliptic case, though the characteristic $CQ_i Q_j$ is not necessarily odd.

For

$$
\begin{aligned}
(CQ_i Q_j &\equiv u^{b_1, a_1} + \ldots + u^{b_p, a_p} + u^{b_1, a} \quad + \ldots + u^{b_k, a}, \quad (0 < k \not> p) \\
&\equiv u^{a_1, a} + \ldots + u^{a_k, a} + u^{b_{k+1}, a_{k+1}} + \ldots + u^{b_p, a_p}, \\
&\equiv u^{a_p, a} - u^{x_1, a_1} - \ldots - u^{x_{p-1}, a_{p-1}}, \\
&\equiv u^{x_1, a_1} + \ldots + u^{x_{p-1}, a_{p-1}} + u^{a, a_p},
\end{aligned}
$$

provided

$$
(x_1, \ldots, x_{p-1}, a) \equiv (a, a, \ldots, a, b_{k+1}, \ldots, b_p),
$$

wherein, in the bracket on the right hand, the number of letters a is k; and is not zero. In other words, by taking $x_1, \ldots, x_{p-1}$ respectively equal to $a_1, \ldots, a, b_{k+1}, \ldots, b_p$—where the number of letters a is $k-1$—we can put the half-period with which the characteristic $CQ_i Q_j$ is associated, into a form differing only by periods from

$$
u^{a_p, a} - u^{x_1, a_1} - \ldots - u^{x_{p-1}, a_{p-1}};
$$

this proves (B. 258) that the function $\vartheta(0 \mid CQ_i Q_j)$ vanishes.

Now, since $2^p + 1$ theta functions of the second order and of characteristic zero are connected by a linear equation, there exists an equation

$$
a\, \vartheta(u+v)\, \vartheta(u-v) = \sum_{\lambda=1}^{s} a_\lambda \vartheta^2(u \mid Q_\lambda),
$$

wherein $a, a_1, \ldots, a_s$ are constants. In this equation increase u by the period $\Omega_C + \Omega_{Q_r}$; then it becomes

$$a \vartheta(u+v \mid C+Q_r)\, \vartheta(u-v \mid C+Q_r) = \sum_{\lambda=1}^{s} a_\lambda e^{-4\pi i (C'+Q'_r) Q_\lambda} \vartheta^2(u \mid C+Q_r+Q_\lambda);$$

therefore, putting $u=0$, we deduce

$$\frac{a_r}{a} = \frac{\vartheta^2(v \mid C+Q_r)\, e^{4\pi i (C'+Q'_r)(C+Q_r)}}{\vartheta^2(0 \mid C+2Q_r)\, e^{-4\pi i (C'+Q'_r) Q_r}},$$

and therefore, *assuming that* $\vartheta(0 \mid u^{b_1, a_1} + \ldots + u^{b_p, a_p})$ *is not zero,*

$$\frac{a_r}{a} = e^{\pi i |C| + \pi i |C, Q_r|} \frac{\vartheta^2(v \mid C+Q_r)}{\vartheta^2(0 \mid C)},$$

so that

$$e^{\pi i |C|} \vartheta^2(0 \mid C)\, \vartheta(u+v)\, \vartheta(u-v) = \sum_{\lambda=1}^{s} e^{\pi i |CQ_r|} \vartheta^2(u \mid Q_r)\, \vartheta^2(v \mid CQ_r),$$

which is the addition equation in question. In regard to the assumption made, the half-period

$$u^{b_1, a_1} + \ldots + u^{b_p, a_p}$$

can be congruent to an expression of the form

$$u^{x_1, a_1} + \ldots + u^{x_{p-1}, a_{p-1}} + u^{a, a_p}$$

only if a be among the set $b_1, \ldots, b_p$; as we have excluded this possibility, the function $\vartheta(0 \mid C)$ does not vanish (B. 260).

In what follows we consider only the case when the branch places $b_1, \ldots, b_p$ are the p places $a_1, \ldots, a_p$; and we take the characteristic A to be zero; then the two results we have obtained are:

(i). Let A_i denote the half integer characteristic associated with the half-period u^{a, a_i}, and consider the 2^p characteristics

$$0,\ A_1, \ldots, A_p,\ A_1A_2, \ldots, A_{p-1}A_p, \ldots, A_1A_2 \ldots A_p;$$

if Q denote any one of these characteristics, other than the first, the function $\vartheta(u \mid Q)$ vanishes for zero values of the argument, though the characteristic Q is not necessarily odd. This result holds for any *hyperelliptic* case.

(ii). We have, in the hyperelliptic case,

$$\vartheta^2(0)\,\vartheta(u+v)\,\vartheta(u-v) = \Sigma\,\vartheta^2(u\,|\,Q)\,\vartheta^2(v\,|\,Q),$$

where the summation extends to the system of 2^p characteristics enumerated in (i).

SECTION XII.

Proof that $\vartheta(u+v)\,\vartheta(u-v)/\vartheta^2(u)\vartheta^2(v)$ is expressible as an integral polynomial in each of the sets of $\frac{1}{2}p(p+1)$ functions

$$\frac{\partial^2}{\partial u_i\,\partial u_j}\log\vartheta(u), \qquad \frac{\partial^2}{\partial v_i\,\partial v_j}\log\vartheta(v),$$

and deduction of a set of 2^p formulæ of this kind.

We put

$$\wp_{ij}(u) = -\frac{\partial^2}{\partial u_i\,\partial u_j}\log\vartheta(u),$$

and, in the equation, proved in Section XI,

$$\vartheta^2(0)\,\vartheta(u+v)\,\vartheta(u-v) = \vartheta^2(u)\,\vartheta^2(v) + \Sigma\,\vartheta_i^2(u)\,\vartheta_i^2(v) + \Sigma\,\vartheta_{ij}^2(u)\,\vartheta_{ij}^2(v) + \Sigma\,\vartheta_{ijk}^2(u)\,\vartheta_{ijk}^2(v) + \ldots\,,$$

wherein $\vartheta_i(u)$ denotes $\vartheta(u\,|\,u^{a_i,\,a_i})$, etc., we suppose $u_1, \ldots, u_p$ to be small, and expand both sides in powers of $u_1, \ldots, u_p$, and then equate coefficients of the same powers of $u_1, \ldots, u_p$, on the two sides, up to the second powers of these quantities. On the right side the functions of three or more suffixes do not give any powers of $u_1, \ldots, u_p$ below the fourth (Section IX), and therefore the coefficients in the expansion of such functions do not enter into the resulting equations. Supposing that

$$\frac{\vartheta(u)}{\vartheta(0)} = 1 + \tfrac{1}{2}(c_{11}u_1^2 + \ldots + 2c_{12}u_1u_2 + \ldots) + \ldots\,,$$

$$\frac{\vartheta_i(u)}{\vartheta(0)} = A_i u_1 + \ldots + P_i u_p + \ldots\,, \qquad (i, j = 1, 2, \ldots, p)$$

$$\frac{\vartheta_{ij}(u)}{\vartheta(0)} = A_{ij} u_1 + \ldots + P_{ij} u_p + \ldots\,,$$

the developed equation, as far as it involves second powers of $u_1, \ldots, u_p$, is

$$1 - \Sigma\Sigma\, u_i u_j \wp_{ij}(v) = 1 + \Sigma\Sigma c_{ij} u_i u_j$$
$$+ \Sigma (A_i u_1 + \ldots + P_i u_p)^2 \frac{\vartheta_i^2(v)}{\vartheta^2(v)} + \Sigma (A_{ij} u_1 + \ldots + P_{ij} u_p)^2 \frac{\vartheta_{ij}^2(v)}{\vartheta^2(v)},$$

and by equating the coefficients we have the following $\frac{1}{2}p(p+1)$ equations for the determination of the $\frac{1}{2}p(p+1)$ quotients

$$q_i^2(v) = \vartheta_i^2(v)/\vartheta^2(v), \quad q_{ij}^2(v) = \vartheta_{ij}^2(v)/\vartheta^2(v), \qquad (i, j = 1, 2, \ldots, p)$$

as integral polynomials in the quantities $\wp_{ij}(v)$,

$$\begin{aligned}
-c_{11} \quad -\wp_{11}(v) &= \sum_i A_i^2 q_i^2(v) + \sum_{i,j}\sum A_{ij}^2 q_{ij}^2(v), \\
-c_{22} \quad -\wp_{22}(v) &= \sum_i B_i^2 q_i^2(v) + \sum_{i,j}\sum B_{ij}^2 q_{ij}^2(v), \\
-c_{12} \quad -\wp_{12}(v) &= \sum_i A_i B_i q_i^2(v) + \sum_{i,j}\sum A_{ij} B_{ij} q_{ij}^2(v), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
-c_{pp} \quad -\wp_{pp}(v) &= \sum_i P_i^2 q_i^2(v) + \sum_{i,j}\sum P_{ij}^2 q_{ij}^2(v), \\
-c_{p,p-1} - \wp_{p,p-1}(v) &= \sum_i P_i N_i q_i^2(v) + \sum_{i,j}\sum P_{ij} N_{ij} q_{ij}^2(v); \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
-c_{p,1} \quad -\wp_{p,1}(v) &= \sum_i P_i A_i q_i^2(v) + \sum_{i,j}\sum P_{ij} A_{ij} q_{ij}^2(v);
\end{aligned}$$

the solution of these equations is given in the next section.

Substituting in the addition equation of Section XI the values for $q_i^2(u)$, $q_i^2(v)$, $q_{ij}^2(u)$, $q_{ij}^2(v)$, which result from these equations, it takes the form

$$\frac{\vartheta^2(0)\,\vartheta(u+v)\,\vartheta(u-v)}{\vartheta^2(u)\,\vartheta^2(v)} = 1 + (\wp_{ij}(u),\ \wp_{ij}(v))$$
$$+ \Sigma \frac{\vartheta_{ijk}^2(u)}{\vartheta^2(u)} \cdot \frac{\vartheta_{ijk}^2(v)}{\vartheta^2(v)} + \ldots;$$

wherein $(\wp_{ij}(u), \wp_{ij}(v))$ denotes an integral polynomial, linear in each of the two sets of $\frac{1}{2}p(p+1)$ functions $\wp_{ij}(u)$, $\wp_{ij}(v)$; but, now, it follows from the formulæ (Section X)

$$\begin{aligned}
\vartheta^{2(n-1)}(u)\,\vartheta_{12\ldots 2n}^2(u) &= \vartheta_{12}^2\,\vartheta_{34}^2 \ldots \vartheta_{2n-1\,2n}^2 \pm \ldots, \\
\vartheta^{2n}(u)\quad \vartheta_{13\ldots 2n+1}^2(u) &= \vartheta_1^2\,\vartheta_{23}^2 \ldots \vartheta_{2n,\,2n+1}^2 \pm \ldots,
\end{aligned}$$

that the quotients $q^2_{ijk\,\ldots}(u)$, of three or more suffixes, are expressible as integral polynomials in the quotients $q^2_i(u)$, $q^2_{ij}(u)$ of one and two suffixes, and therefore as integral polynomials in the $\frac{1}{2}p(p+1)$ functions $\wp_{ij}(u)$.

Hence *the function* $\vartheta(u+v)\,\vartheta(u-v)/\vartheta^2(u)\,\vartheta^2(v)$ *is expressible as an integral polynomial in the* $p(p+1)$ *quantities* $\wp_{ij}(u)$, $\wp_{ij}(v)$.

We pass to a further consequence. Suppose that in the addition formula of Section XI we increase the argument u by Ω_Q, where Q is the half integer characteristic associated with a half-period of the form

$$u^{a,\,a_1}+\ldots\ldots+u^{a,\,a_k};$$

then the formula becomes

$$\vartheta^2(0)\,\vartheta(u+v\,|\,Q)\,\vartheta(u-v\,|\,Q)=\vartheta^2(u\,|\,Q)\,\vartheta^2(v)+\Sigma\vartheta^2(u\,|\,QQ_i)\,\vartheta^2(v\,|\,Q_i),$$

where Q_i becomes in turn every one of the 2^p-1 characteristics formed by combinations of $1, 2, \ldots, p$ of the fundamental characteristics $A_1, \ldots, A_p$; hence

$$\frac{\vartheta^2(0)\,\vartheta(u+v\,|\,Q)\,\vartheta(u-v\,|\,Q)}{\vartheta^2(u\,|\,Q)\,\vartheta^2(v\,|\,Q)}=\frac{\vartheta^2(v)}{\vartheta^2(v\,|\,Q)}+\Sigma\,\frac{\vartheta^2(u\,|\,QQ_i)}{\vartheta^2(u\,|\,Q)}\cdot\frac{\vartheta^2(v\,|\,Q_i)}{\vartheta^2(v\,|\,Q)};$$

here, on the right side, every theta quotient which occurs is of one of the forms

$$\frac{\vartheta^2(u\,|\,QQ_i)}{\vartheta^2(u\,|\,Q)},\quad\frac{\vartheta^2(v\,|\,QQ_i)}{\vartheta^2(u\,|\,Q)},$$

where Q_i is one of the group of 2^p characteristics previously considered; now we have shown that every quotient $\vartheta^2(u\,|\,Q_i)/\vartheta^2(u)$ is expressible as an integral polynomial in the functions $\partial^2\log\vartheta(u)/\partial u_i\,\partial u_j$; it follows that every quotient $\vartheta^2(u\,|\,QQ_i)/\vartheta^2(u\,|\,Q)$ is expressible as an integral polynomial in the functions

$$\wp_{ij}(u\,|\,Q)=-\frac{\partial^2}{\partial u_i\,\partial u_j}\log\vartheta(u\,|\,Q);$$

we have therefore the result:

Let Q *be any one of the group of* 2^p *characteristics*

$$0,\ A_1,\ldots,A_p,\ A_1A_2,\ldots,A_1A_2\ldots A_k,\ldots,$$

before considered; let $\wp_{ij}(u\,|\,Q)=-\partial^2\log\vartheta(u\,|\,Q)/\partial u_i\,\partial u_j$; *then the function*

$$\frac{\vartheta(u+v\,|\,Q)\,\vartheta(u-v\,|\,Q)}{\vartheta^2(u\,|\,Q)\,\vartheta^2(v\,|\,Q)}$$

is expressible as an integral polynomial in the $p(p+1)$ *functions* $\wp_{ij}(u \mid Q)$, $\wp_{ij}(v \mid Q)$.

SECTION XIII.

Explicit determination of the $\frac{1}{2}p(p+1)$ *theta quotients* $q_i^2(u)$, $q_{ij}^2(u)$, *in terms of functions* $\wp_{ij}(u)$.

To solve the equations of Section XII we notice that, as follows from Section VIII, the dissection of the Riemann surface being (I), we have

$$\begin{aligned} &A_r + a_i B_r + \ldots + a_i^{p-1} P_r \\ &\qquad = \frac{1}{\lambda_r}\left[\chi_{p-1}(a_r) + a_i \chi_{p-2}(a_r) + \ldots + a_i^{p-2}\chi_1(a_r) + a_i^{p-1}\right], \\ &\qquad = 0, \text{ when } i \neq r, \\ &\qquad = \frac{1}{\lambda_r} P'(a_r) = i\lambda_r \sqrt{f'(a_r)/4}, \text{ when } i = r, \end{aligned}$$

and

$$\begin{aligned} &A_{rs} + a_i B_{rs} + \ldots + a_i^{p-1} P_{rs} \\ &= -\frac{\zeta_{rs}}{\lambda_r \lambda_s}\left[\psi_{p-1}(a_r, a_s) + a_i \psi_{p-2}(a_r, a_s) + \ldots + a_i^{p-2}\psi_1(a_r, a_s) + a_i^{p-1}(0)\right], \\ &= 0, \text{ when } i \neq r,\ i \neq s, \\ &= -\frac{\zeta_{rs}}{\lambda_r \lambda_s} \frac{P'(a_r)}{a_r - a_s}, \text{ when } i = r,\ r \neq s; \end{aligned}$$

therefore, taking the last p of the equations before given (p. 369), and remembering that $P_{rs} = 0$ for every value of r and s, we have

$$\begin{aligned} &-\left\{\left[c_{pp} + \wp_{pp}(u)\right] a_r^{p-1} + \left[c_{p\,p-1} + \wp_{p,p-1}(u)\right] a_r^{p-2} + \ldots + \left[c_{p1} + \wp_{p,1}(u)\right]\right\} \\ &= P_r i\lambda_r \sqrt{f'(a_r)/4}\, q_r^2(u) \\ &= i\sqrt{f'(a_r)/4}\,.\, q_r^2(u); \end{aligned}$$

also, from the same equations,

$$-\sum_{i=1}^{p}\sum_{j=1}^{p}[c_{ij}+\wp_{ij}(u)]\,a_r^{i-1}a_s^{j-1}$$

$$=\sum_{i=1}^{p}[A_i+a_rB_i+\ldots\ldots+a_r^{p-1}P_i][A_i+a_sB_i+\ldots\ldots+a_s^{p-1}P_i]\,q_i^2(u)$$

$$+\sum_{i=1}^{p}\sum_{j=1}^{p}[A_{ij}+a_rB_{ij}+\ldots\ldots+a_r^{p-1}P_{ij}][A_{ij}+a_sB_{ij}+\ldots\ldots+a_s^{p-1}P_{ij}]\,q_{ij}^2(u),$$

$$=-\frac{\zeta_{rs}^2}{(a_r-a_s)^2}\,\frac{P'(a_r)\,P'(a_s)}{\lambda_r^2\lambda_s^2}\,q_{r,s}^2(u),$$

$$=\frac{\sqrt{f'(a_r)/4}\sqrt{f'(a_s)/4}}{a_r\sim a_s}\,q_{r,s}^2(u),$$

where $a_r\sim a_s,=\binom{A_r}{A_s}(a_r-a_s)$, denotes the difference so chosen that a_s, a_r are in ascending order.

If then we write $M_r=i\sqrt{f'(a_r)/4}$, so that $P'(a_r)=\lambda_r^2M_r$, we have

$$\frac{\vartheta^2(u\,|\,u^{a,\,a_r})}{\vartheta^2(u)}=-\frac{1}{M_r}\sum_{i=1}^{p}[c_{p,i}+\wp_{p,i}(u)]\,a_r^{i-1},$$

$$\frac{\vartheta^2(u\,|\,u^{a,\,a_r}+u^{a,\,a_s})}{\vartheta^2(u)}=\frac{\binom{A_r}{A_s}(a_r-a_s)}{M_rM_s}\sum_{i=1}^{p}\sum_{j=1}^{p}[c_{i,j}+\wp_{i,j}(u)]\,a_r^{i-1}a_s^{j-1};$$

or, using the sigma functions (Section IX),

$$\frac{\sigma_r^2(u)}{\sigma^2(u)}=\frac{1}{Q(a_r)}\sum_{i=1}^{p}[c_{p,i}+\wp_{p,i}(u)]\,a_r^{i-1},$$

$$\frac{\sigma_{rs}^2(u)}{\sigma^2(u)}=\frac{1}{Q(a_r)\,Q(a_s)}\sum_{i=1}^{p}\sum_{j=1}^{p}[c_{i,j}+\wp_{i,j}(u)]\,a_r^{i-1}a_s^{j-1},$$

where $Q(x)=(x-c_1)\ldots\ldots(x-c_p)(x-c)$.

It is to be noticed that if instead of the function $\vartheta(u)$ we use $\overline{\vartheta}(u)$, $=e^{\frac{1}{2}Cu^2}\vartheta(u)$, where Cu^2 denotes a quadratic form in $u_1,\ldots\ldots,u_p$, $=\sum_{i=1}^{p}\sum_{j=1}^{p}C_{ij}u_iu_j$,

then the quadratic form $\frac{1}{2}cu^2$ occurring in the expansion, $\frac{\vartheta(u)}{\vartheta(0)} = 1 + \frac{1}{2}cu^2 + \ldots$, is increased by $\frac{1}{2}Cu^2$, and the function $\wp_{ij}(u), = -\partial^2 \log \vartheta(u)/\partial u_i \partial u_j$ is diminished by C_{ij}. In other words, the quantities $c_{ij} + \wp_{ij}(u)$ are independent of the particular exponential by which we pass from the functions $\Theta\left(u \,\middle|\, \frac{1}{2}\binom{q'}{q}\right)$ to the functions $\vartheta\left(u \,\middle|\, \frac{1}{2}\binom{q'}{q}\right), = e^{au^2}\Theta\left(u \,\middle|\, \frac{1}{2}\binom{q'}{q}\right)$. This is as it should be, because the quotients $q_r(u)$, $q_{rs}(u)$ are equally independent of the exponential factor e^{au^2}.

The quantities a_{ij} occurring in the exponential e^{au^2}, are such that

$$\int_a^x \int_c^z \frac{dx\,dz}{ys} \, \frac{2ys + F(x, z)}{4(x-z)^2} = \prod\nolimits_{z,\,c}^{x,\,a} - 2\sum_{i=1}^{p} \sum_{j=1}^{p} a_{ij} u_i^{x,\,a} u_j^{z,\,c},$$

where $u_1^{x,\,a}, \ldots, u_p^{x,\,a}$ are the fundamental integrals of the first kind, $\prod_{z,\,c}^{x,\,a}$ is Riemann's normal elementary integral of the third kind, and $F(x, z)$ is a rational integral expression, symmetrical in x and z, and of order $p+1$ in each, which satisfies the equations

$$F(z, z) = 2f(z), \quad \left[\frac{\partial}{\partial x} F(x, z)\right]_{x=z} = \frac{d}{dz} f(z),$$

(cf. B. 194, 315); conversely, any rational integral expression in x and z, of order $p+1$ in each, which satisfies the equations just written, may be used in the integral

$$\int_a^x \int_c^z \frac{dx\,dz}{ys} \, \frac{2ys + F(x, z)}{4(x-z)^2}$$

and will give rise to corresponding values for the quantities a_{ij}; for the difference of two such expressions as $F(x, z)$ is necessarily of the form

$$(x-z)^2 [x, z]_{p-1},$$

where $[x, z]_{p-1}$ denotes a rational integral expression symmetrical in x, z and of order $p-1$ in each. Particular forms of such an expression $F(x, z)$ are

(α) if

$$f(x) = \lambda + \lambda_1 x + \lambda_2 x^2 + \ldots + \lambda_{2p+1} x^{2p+1} + \lambda_{2p+2} x^{2p+2},$$

$$F(x, z) = \sum_{i=0}^{p+1} x^i z^i \left[2\lambda_{2i} + \lambda_{2i+1}(x+z)\right];$$

(β) if

$$f(x) = a_x^{2p+2},$$
$$F(x, z) = 2a_x^{p+1} a_z^{p+1};$$

(γ)

$$F(x, z) = 4\left[P(x)\,Q(z) + P(z)\,Q(x)\right].$$

When the expression $F(x, z)$ is once made precise, or what is the same thing, when the quantities a_{ij} are given, it should be possible to determine the coefficients c_{ij} occurring in the expansion

$$\frac{\mathfrak{S}(u)}{\mathfrak{S}(0)} = 1 + \tfrac{1}{2}(c_{11}u_1^2 + \ldots + 2c_{12}u_1u_2 + \ldots) + \ldots;$$

we proceed to prove in fact that these coefficients are given by the equation

$$\Sigma\Sigma c_{ij}u_iu_j = \frac{4\left[P(\xi_1)\,Q(\xi_2) + P(\xi_2)\,Q(\xi_1)\right] - F(\xi_1, \xi_2)}{4(\xi_1 - \xi_2)^2},$$

where, on the right hand, after the division has been carried out, as is always possible, we are to put

$$\xi_1^{i-1} = u_i, \quad \xi_1^{(0)} = u_1, \quad \xi_2^{i-1} = u_i, \quad \xi_2^{(0)} = u_1.$$

For this we may employ the formula

$$\sum_{i=1}^{p}\sum_{j=1}^{p} \wp_{ij}(u^{x,a} + u^{x_1,a_1} + \ldots + u^{x_p,a}) \,.\, x_r^{i-1}x_s^{j-1} = \frac{F(x_r, x_s) - 2y_ry_s}{4(x_r - x_s)^2},$$

which is independent of the particular form of the polynomial $F(x, z)$ adopted; this formula is deducible by differentiation from the well-known formula

$$\sum_{i=1}^{p}\Pi_{z_i,\bar{c}_i}^{x,k} = \log\left[\frac{\Theta(u^{x,a} + u^{z_1,a_1} + \ldots + u^{z_p,a_p})}{\Theta(u^{x,a} + u^{c_1,a_1} + \ldots + u^{c_p,a_p})}\Big/\frac{\Theta(u^{k,a} + u^{z_1,a_1} + \ldots + u^{z_p,a_p})}{\Theta(u^{k,a} + u^{c_1,a_1} + \ldots + u^{c_p,a_p})}\right],$$

where $\bar{z}_i$, $\bar{c}_i$ denote the places conjugate to z_i and c_i respectively.

It is easy to see, by differentiation of the equation

$$\frac{\mathfrak{S}(u)}{\mathfrak{S}(0)} = 1 + \tfrac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} c_{ij}u_iu_j + \ldots,$$

that when the arguments are zero,

$$\wp_{ij}(0) = -c_{ij};$$

hence, if in the formula quoted above we put $x = a$, $x_1 = a_1, \ldots, x_p = a_p$, we obtain

$$-\sum_{i=1}^{p}\sum_{j=1}^{p} c_{ij}\, a_r^{i-1} a_s^{j-1} = +\frac{F(a_r, a_s)}{4(a_r - a_s)^2}, \tag{δ}$$

and, similarly, if we put $x = a_s$, $x_1 = a_1, \ldots, x_r = a_r, \ldots, x_s = a, \ldots, x_p = a_p$, we obtain

$$-\sum_{i=1}^{p} c_{p,i}\, a_r^{i-1} = \lim_{x=\infty} \frac{F(x, a_r)}{4x^{p-1}(x - a_r)^2}, \tag{ϵ}$$

from the $\frac{1}{2}p(p-1) + p = \frac{1}{2}p(p+1)$ linear equations (δ), (ϵ), we can determine the $\frac{1}{2}p(p+1)$ quantities c_{ij}; in order then to prove the formula

$$\sum_{i=1}^{p}\sum_{j=1}^{p} c_{ij}\, x^{i-1} z^{j-1} = \frac{4[P(x)\,Q(z) + P(z)\,Q(x)] - F(x, z)}{4(x - z)^2},$$

it is sufficient to prove that it includes the equation (δ), (ϵ). This is immediately obvious.

In particular when, for

$$f(x) = \lambda + \lambda_1 x + \ldots + \lambda_{2p} x^{2p} + 4x^{2p+1},$$

we take

$$F(x, z) = \sum_{i=0}^{p+1} x^i z^i\,[2\lambda_{2i} + \lambda_{2i+1}(x + z)],$$

we find

$$\sum_{j=1}^{p} c_{p,j}\, z^{j-1} = P(z) - z^p,$$

and therefore, if

$$P(x) = x^p + d_1 x^{p-1} + d_2 x^{p-2} + \ldots + d_p,$$

we have

$$c_{p,i} = d_{p-i+1}.$$

It follows from these results that the formulæ put down at the commencement of this section can be written

$$-M_r\frac{\vartheta^2(u\,|\,u^{a,\,a_r})}{\vartheta^2(u)} = Q(a_r)\frac{\sigma_r^2(u)}{\sigma^2(u)} = \sum_{i=1}^{p} a_r^{i-1}\wp_{p,\,i}(u) - \lim_{x=\infty}\frac{F(x,\,a_r)}{4x^{p-1}(x-a_r)^2},$$

$$\frac{M_r M_s}{\binom{A_r}{A_s}(a_r-a_s)}\cdot\frac{\vartheta^2(u\,|\,u^{a,\,a_r}+u^{a,\,a_s})}{\vartheta^2(u)} = Q(a_r)\,Q(a_s)\frac{\sigma_{rs}^2(u)}{\sigma^2(u)}$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p} a_r^{i-1}a_s^{j-1}\wp_{ij}(u) - \frac{F(a_r,\,a_s)}{4(a_r-a_s)^2};$$

in the second of these the right side is (Section VI) equal to

$$M_r M_s\frac{\partial^2}{\partial V_r\,\partial V_s}\log\vartheta(u) - \frac{F(a_r,\,a_s)}{4(a_r-a_s)^2};$$

from the first of these we can deduce the equation satisfied by the values of $x_1, \ldots, x_p$ which satisfy the inversion equation

$$u_i = u_i^{x_1,\,a} + \ldots + u_i^{x_p,\,a_p}; \qquad (i = 1,\,2,\,\ldots,\,p)$$

for let the coefficient of x^{p+1} in $F(x,\,z)$ be

$$T(z) = t_0z^{p+1} + t_1z^p + t_2z^{p-1} + \ldots + t_{p+1};$$

then we have

$$\lim_{x=\infty}\frac{F(x,\,a_r)}{4x^{p-1}(x-a_r)^2} = \tfrac{1}{4}\,T(a_r);$$

further

$$\frac{T(x)}{P(x)} = t_0x + A + \sum_{r=1}^{p}\frac{T(a_r)}{(x-a_r)P'(a_r)},$$

where $A = t_1 - t_0d_1$,

and therefore

$$\sum_{r=1}^{p}\frac{P(x)\,T(a_r)}{(x-a_r)P'(a_r)} = T(x) - (t_0x + A)\,P(x);$$

hence $(x-x_1)\ldots(x-x_p),\ = F(x),$

$$= P(x) + \sum_{r=1}^{p}\frac{P(x)\,F(a_r)}{(x-a_1)\,P'(a_r)},$$

is (by Section VI) equal to

$$P(x)+\sum_{r=1}^{p}\frac{P(x)\,M_r}{(x-a_r)\,P'(a_r)}\,\frac{\vartheta^2(u\,|\,u^{a,\,a_r})}{\vartheta^2(u)},$$

$$=P(x)-\sum_{r=1}^{p}\frac{P(x)}{(x-a_r)\,P'(a_r)}\Big[\sum_{i=1}^{p}a_r^{i-1}\wp_{p,\,i}(u)-\tfrac{1}{4}T(a_r)\Big];$$

$$=P(x)+\tfrac{1}{4}\,T(x)-\tfrac{1}{4}(t_0x+A)\,P(x)-\sum_{i=1}^{p}x^{i-1}\wp_{p,\,i}(u);$$

thus the equation for $x_1, \ldots, x_p$ is

$$\tfrac{1}{4}\,T(x)-\sum_{i=1}^{p}x^{i-1}\wp_{p,\,i}(u)=\tfrac{1}{4}(t_0x+A-4)\,P(x);$$

in particular when

$$F(x,\,z)=\sum_{i=0}^{p+1}x^iz^i\,[2\lambda_{2i}+\lambda_{2i+1}(x+z)],$$

and $\lambda_{2p+2}=0$, $\lambda_{2p+1}=4$, we have $T(x)=4x^p$, $t_0=0$, $t_1=4$, $A=4$, and the equation becomes

$$x^p-x^{p-1}\wp_{p,\,p}(u)-x^{p-2}\wp_{p,\,p-1}(u)-\ldots-\wp_{p,\,1}(u)=0.$$

(Cf. Bolza, American Journal, XVII (1895), and B. 324.)

For the sake of completeness we put here also a particular case of a formula given by Bolza (Göttingen, Nachrichten, 1894); from the $\frac{1}{2}p(p+1)$ equations expressed by

$$\sum_{i=1}^{p}\sum_{j=1}^{p}\wp_{ij}(u)\,x_r^{i-1}x_s^{j-1}=\frac{F(x_r,\,x_s)-2y_r\,y_s}{4\,(x_r-x_s)^2},$$

$$\sum_{i=1}^{p}\wp_{p,\,i}(u)\,x_r^{i-1}\qquad=\lim_{x=\infty}\Big[\frac{F(x,\,x_r)}{4x^{p-1}(x-x_r)^2}\Big],$$

where $$u_i=u_i^{x_1,\,a_1}+\ldots+u_i^{x_p,\,a_p},\qquad (i=1,\,2,\ldots,\,p)$$

and x_r, x_s are chosen from $x_1, \ldots, x_p$, we can obtain the algebraic expressions

of the $\frac{1}{2}p(p+1)$ quantities $\wp_{ij}(u)$; the result is that, *for any values of x and z*, we have, with $F(x)=(x-x_1)\ldots.(x-x_p)$,

$$\sum_{i=1}^{p}\sum_{j=1}^{p}[\wp_{ij}(u)+c_{ij}]\,x^{i-1}z^{j-1}=\tfrac{1}{4}F(x)F(z)\Big[\sum_{i=1}^{p}\frac{y_i}{(x_i-x)(x_i-z)F'(x_i)}\Big]^2$$
$$-\frac{[F(x)P(z)-F(z)P(x)][F(x)Q(z)-F(z)Q(x)]}{(x-z)^2F(x)F(z)};$$

to prove this it is sufficient to see that this last result includes the $\frac{1}{2}p(p+1)$ equations before given. Now putting x_r, x_s for x, z, the right side of the formula last written becomes

$$\tfrac{1}{4}F(x)F(z)\Big[\frac{y_r}{(x_r-x)(x_r-z)F'(x_r)}+\frac{y_s}{(x_s-x)(x_s-z)F'(x_s)}\Big]^2,$$
$$-\tfrac{1}{4}\frac{f(z)}{(x-z)^2}\frac{F(x)}{F(z)}-\tfrac{1}{4}\frac{f(x)}{(x-z)^2}\frac{F(z)}{F(x)}+\frac{P(x)Q(z)+P(z)Q(x)}{(x-z)^2},$$
$$=\frac{4[P(x_r)Q(x_s)+P(x_s)Q(x_r)]-2y_ry_s}{4(x_r-x_s)^2},$$
$$=\sum_{i=1}^{p}\sum_{j=1}^{p}c_{ij}x_r^{i-1}x_s^{j-1}+\frac{F(x_r,x_s)-2y_ry_s}{4(x_r-x_s)^2},$$

which is in accordance with $\frac{1}{2}p(p-1)$ of the equations in question; while, dividing by x^{p-1} and putting $z=x_r$, we obtain

$$\sum_{i=1}^{p}[\wp_{p,i}(u)+c_{p,i}]\,x_r^{i-1}=P(x_r)=\lim_{x=\infty}\frac{P(x)Q(x_r)+P(x_r)Q(x)}{(x-x_r)^2},$$
$$=\sum_{i=1}^{p}c_{p,i}x_r^{i-1}+\lim_{x=\infty}\frac{F(x,x_r)}{4x^{p-1}(x-x_r)^2},$$

which is in accordance with the last p equations in question.

In conclusion we calculate the values of c_{11}, c_{12}, c_{22} for $p=2$, with various hypotheses in regard to $F(x,z)$.

When

$$f(x) = \lambda + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4 + 4x^5$$

$$F(x, z) = \sum_{=0}^{p+1} x^i z^i [2\lambda_{2i} + \lambda_{2i+1}(x+z)]$$

$$= 2\lambda + \lambda_1(x+z) + xz[2\lambda_2 + \lambda_3(x+z)] + x^2z^2[2\lambda_4 + 4(x+z)],$$

we have already found

$$c_{2,1} = c_{1,2} = a_1 a_2, \quad c_{2,2} = -a_1 - a_2;$$

thence from

$$c_{11} + c_{12}(a_1 + a_2) + c_{22} a_1 a_2 = -\frac{F(a_1, a_2)}{4(a_1 - a_2)^2},$$

we have

$$c_{11} = -\frac{F(a_1, a_2)}{4(a_1 - a_2)^2};$$

now

$$F(x, z) - f(x) - f(z)$$

$$= (x-z)^2 [-\lambda_2 - \lambda_3(x+z) - \lambda_4(x+z)^2 - 4(x+z)(x^2 + xz + z^2)],$$

and therefore

$$4c_{11} = \lambda_2 + \lambda_3(a_1 + a_2) + \lambda_4(a_1 + a_2)^2 + 4(a_1 + a_2)(a_1^2 + a_1 a_2 + a_2^2),$$

or, if we write

$$f(x) = 4(x^5 + 5A_1 x^4 + 10A_2 x^3 + 10A_3 x^2 + 5A_4 x + A_5),$$

then

$$c_{11} = 10A_3 + 10A_2(a_1 + a_2) + 5A_1(a_1 + a_2)^2$$
$$+ (a_1 + a_2)(a_1^2 + a_1 a_2 + a_2^2).$$

If, however, with

$$f(x) = a_x^6 = 4p_x^3 q_x^3,$$

we take

$$F(x, z) = 2a_x^3 a_z^3,$$

then

$$c_{11} + c_{12}(x+z) + c_{22}xz,$$

$$= \frac{p_x^3 q_z^3 + p_z^3 q_x^3 - \frac{1}{2} a_x^3 a_z^3}{(x-z)^2},$$

$$= \tfrac{9}{10}(pq)^2 (p_x q_z + p_z q_x),$$

as can easily be calculated. Hence, putting

$$f(x) = 4(x^5 + 5A_1 x^4 + 10A_2 x^3 + 10A_3 x^2 + 5A_4 x + A_5),$$

we find, with $h_1 = -a_1 - a_2$, $h_2 = a_1 a_2$, that

$$c_{22} = h_1 - 2A_1, \quad c_{12} = h_2 - A_2, \quad c_{11} = -h_1^3 + 5h_1^2 A_1 - 10h_1 A_2 + h_1 h_2 + 6A_3;$$

if we denote these by $\bar{c}_{22}$, $\bar{c}_{12}$, $\bar{c}_{11}$, and those just found by c_{22}, c_{12}, c_{11}, we have

$$c_{22} - \bar{c}_{22} = 2A_1, \quad c_{12} - \bar{c}_{12} = A_2, \quad c_{11} - \bar{c}_{11} = 4A_3 \, .$$

In other words, the theta functions formed with $F(x, z)$ equal to the former value are obtained from those formed with $F(x, z) = 2a_x^2 a_z^2$, by multiplying by the factor

$$e^{A_1 u_2^2 + A_2 u_1 u_2 + 2A_3 u_1^2}.$$

SECTION XIV.

Evaluation of $\vartheta(u+v)\vartheta(u-v)/\vartheta^2(u)\vartheta^2(v)$ *in certain cases.*

Putting $\vartheta_{12}(u) = \vartheta(u \,|\, u^{a, a_1} + u^{a, a_2})$ we work out first the case, in which $p = 2$, of

$$\frac{\vartheta_{12}(u+v)\,\vartheta_{12}(u-v)}{\vartheta_{12}^2(u)\,\vartheta_{12}^2(v)}.$$

From the addition formula (Section XI), by adding the half-period $u^{a, a_1} + u^{a, a_2}$ to the argument u, and dividing by $\vartheta_{12}^2(u)\,\vartheta_{12}^2(v)$ we obtain, on the hypothesis of the dissection (I),

$$\frac{\vartheta^2(0)\,\vartheta_{12}(u+v)\,\vartheta_{12}(u-v)}{\vartheta_{12}^2(u)\,\vartheta_{12}^2(v)} = \frac{\theta^2}{\theta_{12}^2} - \frac{\vartheta^2}{\vartheta_{12}^2} - \frac{\vartheta_2^2}{\vartheta_{12}^2}\cdot\frac{\theta_1^2}{\theta_{12}^2} + \frac{\vartheta_1^2}{\vartheta_{12}^2}\cdot\frac{\theta_2^2}{\theta_{12}^2},$$

where $\vartheta, \vartheta_1, \ldots$ denote $\vartheta(u)$, $\vartheta(u \,|\, u^{a, a_1}), \ldots$ and $\theta, \theta_1, \ldots$ denote $\vartheta(v)$, $\vartheta(v \,|\, u^{a, a_1}), \ldots$ Now if L_{ij} be defined by

$$\begin{aligned} L_{ij} &= -\frac{\partial^2}{\partial u_i \, \partial u_j} \log \vartheta_{12}(u) + c_{ij}, \\ &= \wp_{ij}(u \,|\, u^{a, a_1} + u^{a, a_2}) + c_{ij}, \end{aligned}$$

we have (Section XIII)

$$-\frac{\vartheta_2^2}{\vartheta_{12}^2} = -\frac{1}{M_1}(L_{22} a_1 + L_{21}), \quad \frac{\vartheta_1^2}{\vartheta_{12}^2} = -\frac{1}{M_2}(L_{22} a_2 + L_{21}),$$

$$-\frac{\vartheta^2}{\vartheta_{12}^2} = \frac{a_2 - a_1}{M_1 M_2}\left[L_{22} a_1 a_2 + L_{21}(a_1 + a_2) + L_{11}\right];$$

hence

$$\frac{M_1 M_2}{a_2 - a_1} \frac{\vartheta^2(0)\,\vartheta_{12}(u+v)\,\vartheta_{12}(u-v)}{\vartheta_{12}^2(u)\,\vartheta_{12}^2(v)}$$

$$= (L_{22} - L'_{22})\,a_1 a_2 + (L_{21} - L'_{21})(a_1 + a_2) + L_{11} - L'_{11}$$
$$+ \frac{1}{a_2 - a_1}\left[(L_{22}\,a_1 + L_{21})(L'_{22}\,a_2 + L'_{21}) - (L'_{22}\,a_1 + L'_{21})(L_{22}\,a_2 + L_{21})\right],$$

where L'_{ij} is the same function of v that L_{ij} is of u; and if we use $\wp_{ij}$ to denote $\wp_{ij}(u\,|\,u^{a,\,a_1} + u^{a,\,a_2})$, and $\wp'_{ij}$ to denote $\wp_{ij}(v\,|\,u^{a,\,a_1} + u^{a,\,a_2})$, this gives

$$(\wp_{22} - \wp'_{22})\,a_1 a_2 + (\wp_{21} - \wp'_{21})(a_1 + a_2) + \wp_{11} - \wp'_{11}$$
$$+ (\wp_{21} + c_{21})(\wp'_{22} + c_{22}) - (\wp'_{21} + c_{21})(\wp_{22} + c_{22}),$$
$$= \wp_{11} - \wp'_{11} + (\wp_{21} - \wp'_{21})(a_1 + a_2 + c_{22}) + (\wp_{22} - \wp'_{22})(a_1 a_2 - c_{21}) + \wp_{21}\wp'_{22} - \wp_{22}\wp'_{21}.$$

Recalling the definition of the σ-function (Section IX), we therefore have the result

$$-\frac{\sigma_{12}(u+v)\,\sigma_{12}(u-v)}{\sigma_{12}^2(u)\,\sigma_{12}^2(v)} = \wp_{11}(u) - \wp_{11}(v) + [\wp_{21}(u) - \wp_{21}(v)](a_1 + a_2 + c_{22})$$
$$+ [\wp_{22}(u) - \wp_{22}(v)][a_1 a_2 - c_{21}]$$
$$+ \wp_{21}(u)\,\wp_{22}(v) - \wp_{22}(u)\,\wp_{21}(v).$$

Hence, if

$$f(x) = \lambda + \lambda_1 x + \lambda_2 x^2 + \ldots + \lambda_4 x^4 + 4x^5,$$
$$F(x, z) = \sum_{i=0}^{p} x^i z^i\,[2\lambda_{2i} + \lambda_{2i+1}(x + z)],$$

we have (Section XIII)

$$-\frac{\sigma_{12}(u+v)\,\sigma_{12}(u-v)}{\sigma_{12}^2(u)\,\sigma_{12}^2(v)} = \wp_{11}(u) - \wp_{11}(v) + \wp_{21}(u)\,\wp_{22}(v) - \wp_{22}(u)\,\wp_{21}(v);$$

while if

$$f(x) = 4\,(x^5 + 5A_1 x^4 + 10A_2 x^3 + 10A_3 x^2 + 5A_4 x + A_5)$$
$$= a_x^6,$$

and $\quad F(x, z) = 2a_x^3 a_z^3,$

we similarly have

$$-\frac{\sigma_{12}(u+v)\,\sigma_{12}(u-v)}{\sigma_{12}^2(u)\,\sigma_{12}^2(v)} = \wp_{11}(u) - \wp_{11}(v) - 2A_1\,[\wp_{12}(u) - \wp_{12}(v)]$$
$$+ A_2\,[\wp_{22}(u) - \wp_{22}(v)] + \wp_{21}(u)\,\wp_{22}(v) - \wp_{21}(v)\,\wp_{22}(u).$$

In either of these the function $\wp_{ij}(u)$ is given by

$$\wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma_{12}(u).$$

The forms here obtained for the right sides in these equations can be variously modified by utilizing the fact that the three functions $\wp_{11}(u)$, $\wp_{12}(u)$, $\wp_{22}(u)$ are connected by an algebraic equation.

Consider next the case, for $p = 3$, of the function

$$\frac{\vartheta_{123}(u+v)\,\vartheta_{123}(u-v)}{\vartheta_{123}^2(u)\,\vartheta_{123}^2(v)},$$

where $\vartheta_{123}(u) = \vartheta(u \mid u^{a,a_1} + u^{a,a_2} + u^{a,a_3})$. We put, in the course of the work, ϑ_{123} for $\vartheta_{123}(u)$ and θ_{123} for $\vartheta_{123}(v)$, etc.; further, we put

$$\wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \partial u_j} \log \vartheta_{123}(u) = \wp_{ij},$$

$$\wp_{ij}(v) = -\frac{\partial^2}{\partial v_i \partial v_j} \log \vartheta_{123}(v) = \wp'_{ij},$$

and

$$A = \wp_{11} - \wp'_{11}, \quad F = \wp_{23} - \wp'_{23},$$
$$B = \wp_{22} - \wp'_{22}, \quad G = \wp_{31} - \wp'_{31},$$
$$C = \wp_{33} - \wp'_{33}, \quad H = \wp_{12} - \wp'_{12}.$$

We have, for the dissection (I),

$$\vartheta^2 \vartheta_{123}^2 = \vartheta_1^2 \vartheta_{23}^2 - \vartheta_2^2 \vartheta_{31}^2 + \vartheta_3^2 \vartheta_{12}^2,$$

and therefore

$$\frac{\vartheta^2}{\vartheta_{123}^2} = \frac{\vartheta_1^2}{\vartheta_{123}^2} \cdot \frac{\vartheta_{23}^2}{\vartheta_{123}^2} - \frac{\vartheta_2^2}{\vartheta_{123}^2} \cdot \frac{\vartheta_{31}^2}{\vartheta_{123}^2} + \frac{\vartheta_3^2}{\vartheta_{123}^2} \cdot \frac{\vartheta_{12}^2}{\vartheta_{123}^2};$$

also, by the addition formula (Section XI), by increasing the argument u by the half-period $u^{a,a_1} + u^{a,a_2} + u^{a,a_3}$, we have

$$\frac{\vartheta^2(0)\,\vartheta_{123}(u+v)\,\vartheta_{123}(u-v)}{\vartheta_{123}^2(u)\,\vartheta_{123}^2(v)}$$
$$= \frac{\theta^2}{\theta_{123}^2} + \frac{\vartheta^2}{\vartheta_{123}^2} - \frac{\vartheta_{23}^2}{\vartheta_{123}^2}\,\frac{\theta_1^2}{\theta_{123}^2} - \frac{\vartheta_1^2}{\vartheta_{123}^2}\,\frac{\theta_{23}^2}{\theta_{123}^2} + \frac{\vartheta_{31}^2}{\vartheta_{123}^2}\,\frac{\theta_2^2}{\theta_{123}^2}$$
$$+ \frac{\vartheta_2^2}{\vartheta_{123}^2}\,\frac{\theta_{31}^2}{\theta_{123}^2} - \frac{\vartheta_{12}^2}{\vartheta_{123}^2}\,\frac{\theta_3^2}{\theta_{123}^2} - \frac{\vartheta_3^2}{\vartheta_{123}^2}\,\frac{\theta_{12}^2}{\theta_{123}^2}$$
$$= \left(\frac{\vartheta_1^2}{\vartheta_{123}^2} - \frac{\theta_1^2}{\theta_{123}^2}\right)\left(\frac{\vartheta_{23}^2}{\vartheta_{123}^2} - \frac{\theta_{23}^2}{\theta_{123}^2}\right) - \left(\frac{\vartheta_2^2}{\vartheta_{123}^2} - \frac{\theta_2^2}{\theta_{123}^2}\right)\left(\frac{\vartheta_{13}^2}{\vartheta_{123}^2} - \frac{\theta_{13}^2}{\theta_{123}^2}\right)$$
$$+ \left(\frac{\vartheta_3^2}{\vartheta_{123}^2} - \frac{\theta_3^2}{\theta_{123}^2}\right)\left(\frac{\vartheta_{12}^2}{\vartheta_{123}^2} - \frac{\theta_{12}^2}{\theta_{123}^2}\right);$$

now we have proved (Section XIII) that

$$\frac{\vartheta_{rs}^2(u)}{\vartheta^2(u)} = \frac{\binom{A_r}{A_s}(a_r - a_s)}{M_r M_s} \sum_{i=1}^{p} \sum_{j=1}^{p} a_r^{i-1} a_s^{j-1} \left[c_{ij} - \frac{\partial^2}{\partial u_i \partial u_j} \log \vartheta(u)\right],$$

$$\frac{\vartheta_2^2(u)}{\vartheta_r^2(u)} = -\frac{1}{M_r} \sum_{i=1}^{p} a_r^{i-1}\left[c_{p,i} - \frac{\partial^2}{\partial u_p \partial u_i} \log \vartheta(u)\right];$$

hence if r, s, t denote $1, 2, 3$, in some order, we have, the function $\wp_{ij}(u)$ being, as before stated, derived from $\vartheta_{123}(u)$,

$$\binom{A_1 A_2 A_3}{A_r A_s} \frac{\vartheta_t^2(u)}{\vartheta_{123}^2(u)} = \frac{\binom{A_r}{A_s}(a_r - a_s)}{M_r M_s} \sum_{i=1}^{p} \sum_{j=1}^{p} a_r^{i-1} a_s^{j-1} [c_{ij} + \wp_{ij}(u)],$$

$$\binom{A_1 A_2 A_3}{A_t} \frac{\vartheta_{rs}^2(u)}{\vartheta_{123}^2(u)} = -\frac{1}{M_t} \sum_{i=1}^{p} a_t^{i-1} [c_{p,i} + \wp_{p,i}(u)];$$

thus, in the notation previously explained,

$$M_1 M_2 M_3 \frac{\vartheta^2(0)\, \vartheta_{123}(u+v)\, \vartheta_{123}(u-v)}{\vartheta_{123}^2(u)\, \vartheta_{123}^2(v)}$$

$$= \Sigma (a_2 - a_3)[a_1^2 C + a_1 F + G][A + B a_2 a_3 + C a_2^2 a_3^2 + F a_2 a_3 (a_2 + a_3) + G(a_2^2 + a_3^2) + H(a_2 + a_3)],$$

where the sign of summation refers to the three suffixes a_1, a_2, a_3, to be taken in cyclical order,

$$= (a_2 - a_3)(a_3 - a_1)(a_1 - a_2)[HF - BG - CA + G^2],$$

as can be immediately verified; now we have seen (Section VII) that

$$\sigma_{123}(u) = \frac{\lambda_1 \lambda_2 \lambda_3}{\zeta_{23} \zeta_{31} \zeta_{12}} \frac{\vartheta_{123}(u)}{\vartheta(0)},$$

where $\qquad \lambda_r^2 M_r = P'(a_r) = (a_r - a_s)(a_r - a_t);$

hence

$$\frac{\sigma_{123}(u+v)\, \sigma_{123}(u-v)}{\sigma_{123}^2(u)\, \sigma_{123}^2(v)} = \frac{(a_3 - a_2)(a_3 - a_1)(a_2 - a_1)}{\lambda_1^2 \lambda_2^2 \lambda_3^2} \frac{\vartheta^2(0)\, \vartheta_{123}(u+v)\, \vartheta_{123}(u-v)}{\vartheta_{123}^2(u)\, \vartheta_{123}^2(v)},$$

$$= -\frac{M_1 M_2 M_3}{(a_2 - a_3)(a_3 - a_1)(a_1 - a_2)} \frac{\vartheta^2(0)\, \vartheta_{123}(u+v)\, \vartheta_{123}(u-v)}{\vartheta_{123}^2(u)\, \vartheta_{123}^2(v)};$$

thus, finally,

$$-\frac{\sigma_{123}(u+v)\,\sigma_{123}(u-v)}{\sigma_{123}^2(u)\,\sigma_{123}^2(v)} = [\wp_{31}(u)-\wp_{31}(v)]^2 - [\wp_{31}(u)-\wp_{31}(v)][\wp_{22}(u)-\wp_{22}(v)]$$
$$-[\wp_{33}(u)-\wp_{33}(v)][\wp_{11}(u)-\wp_{11}(v)]$$
$$+[\wp_{12}(u)-\wp_{12}(v)][\wp_{23}(u)-\wp_{23}(v)].$$

Other cases can be similarly worked out. In accordance however with the results enunciated by Weierstrass, Crelle LXXXIX (1880), every $p+1$ of the functions $\wp_{ij}(u)$ are connected by an algebraical equation, which is in fact not difficult to obtain. It becomes therefore proper to investigate the modifications following from the introduction of these algebraical equations. For the present paper it is sufficient to have investigated a method whereby the actual equations can always be calculated.

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